Log-linear ODEs and Applications to the Ricci Flow for Homogeneous Spaces

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Background

Homogeneous Spaces

A homogeneous space is a Riemannian manifold with an isometry between any two points. This means that all of the geometric information is encoded in the Lie algebra structure of the tangent space at one point. It follows that the geometry is completely determined by a finite set of real numbers, called structure constants.

Ricci Flow

Given a Riemannian manifold (M, g), the Ricci curvature at a point, denoted ric_g, is the sum of the sectional curvatures at that point relative to a fixed direction. A family $(M, g(t))$ of Riemannian manifolds solves the Ricci flow if

$$\frac{\partial}{\partial t} g = -2\text{ric}_{g(t)},$$

where $g(0) = g_0$ for some initial metric g_0. The Ricci flow is a geometric evolution that tries to evenly distribute the Ricci curvature around the manifold.

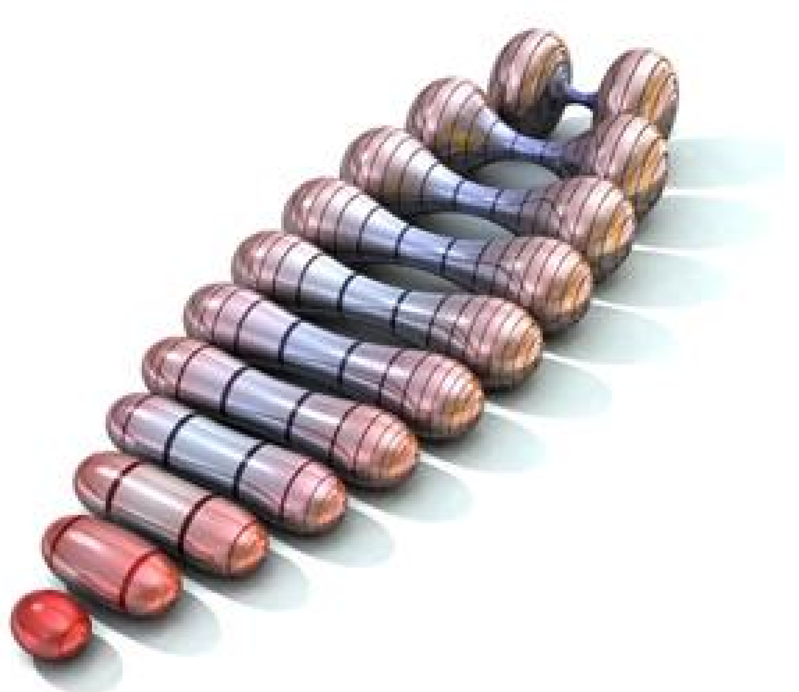

Figure 1: As the geometry of the manifold evolves under the Ricci flow, it becomes more symmetric.

Homogeneous Log-linear ODE

A homogeneous log-linear (HLL) system of ODE is one of the form

$$x_1' = x_1(a_{11}x_1 + \cdots + a_{1n}x_n)$$
$$\vdots$$
$$x_n' = x_n(a_{n1}x_1 + \cdots + a_{nn}x_n).$$

The matrix $A = (a_{ij})$ is called the defining matrix for the system. Although the Ricci flow is in general a PDE, on many homogeneous spaces the flow reduces to a system of HLL ODE. The variables x_i each correspond to a distinct structure constant for the space. As the structure constants evolve, so does the geometry.

Collapsing

When a solution trajectory approaches the origin as $t \to \infty$, the manifold is said to be collapsing under the flow. This corresponds to the space flattening out to look like Euclidean space. **Manifolds will collapse under the flow under any of the following conditions:**

1. All the entries of the defining matrix are negative.
2. The defining matrix A is symmetric and there exists a vector v with positive entries such that Av has negative entries.
3. If A is a 2×2 matrix, then collapsing is equivalent to either having a_{11} and $a_{22} < 0$ and $\det A > 0$, or having all the entries negative.

Existence of Soliton Trajectories

Solitons are special Ricci flow solutions that evolve in a self-similar way. In the phase plane, their trajectories are rays coming in towards the origin. **We completely characterized when soliton trajectories exist.** The ray in the direction of a vector v is a soliton trajectory if and only if

$$Av = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}$$

where A is the defining matrix. Combined with a previous result, this shows that whenever there is a soliton trajectory, the manifolds are also collapsing.

Stability

A solution trajectory is said to be stable if it approaches a soliton trajectory as time increases. One hopes that under the Ricci flow, solution trajectories will approach the origin (collapse) while remaining stable. Given defining matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

we were able to completely determine when trajectories will collapse yet remain stable. **In this case, solutions will remain stable only when $b > a$.**

Future Study

While our results on collapsing and the existence of soliton trajectories are strong, we lack results on stability for dimensions higher than two. Given the nature of our proof for dimension two, a higher dimensional analog seems attainable. As it stands, checking a particular system for stability is often possible, however finding a general condition for stability based on the coefficients of the defining matrix proves more difficult.

Analysis of the 2×2 Symmetric Case

2×2 Symmetric HLL ODE

We began our analysis by considering systems of symmetric HLL ODE of the form

$$x' = x(ax + by)$$
$$y' = y(bx + ay).$$

We classified when collapsing occurs, when solitons exist, and when solutions approach soliton trajectories modulo rescaling. The three cases shown exhaust the possibilities with one exception. If the x and y nullclines are equal, then there will be a line of fixed points and thus neither collapsing nor soliton trajectories occur.

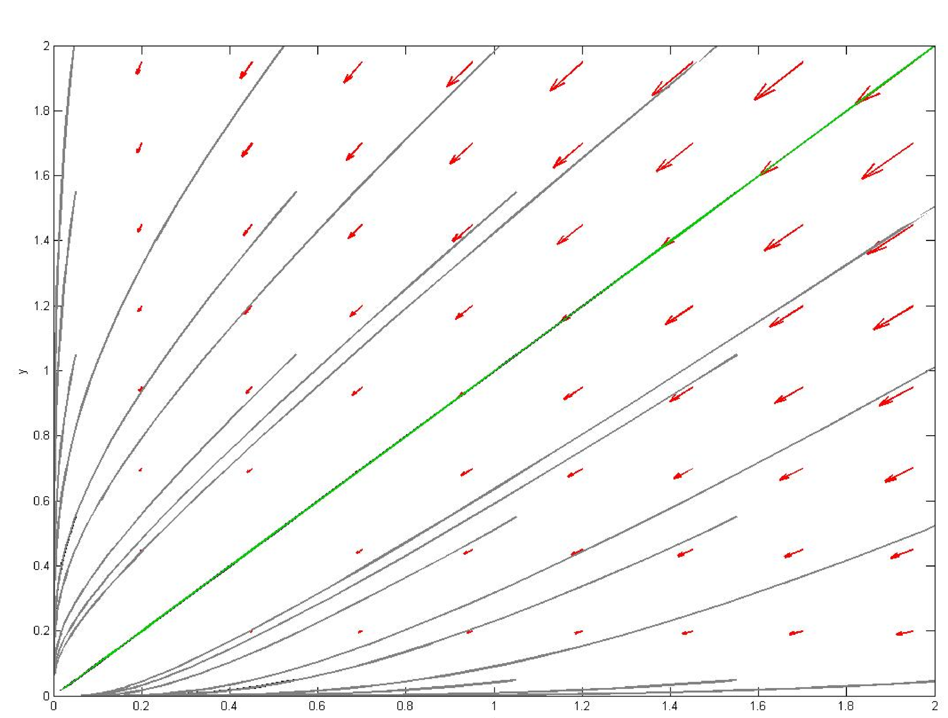

Figure 1: In this case, $a < 0$ and $b < 0$. Solutions are collapsing, but are not approaching the soliton ray.

Possibilities for the Flow

These examples show that the flow does not always do what we would like. Sometimes manifolds do not collapse under the flow, and even when collapsing occurs, trajectories may not approach the soliton ray.

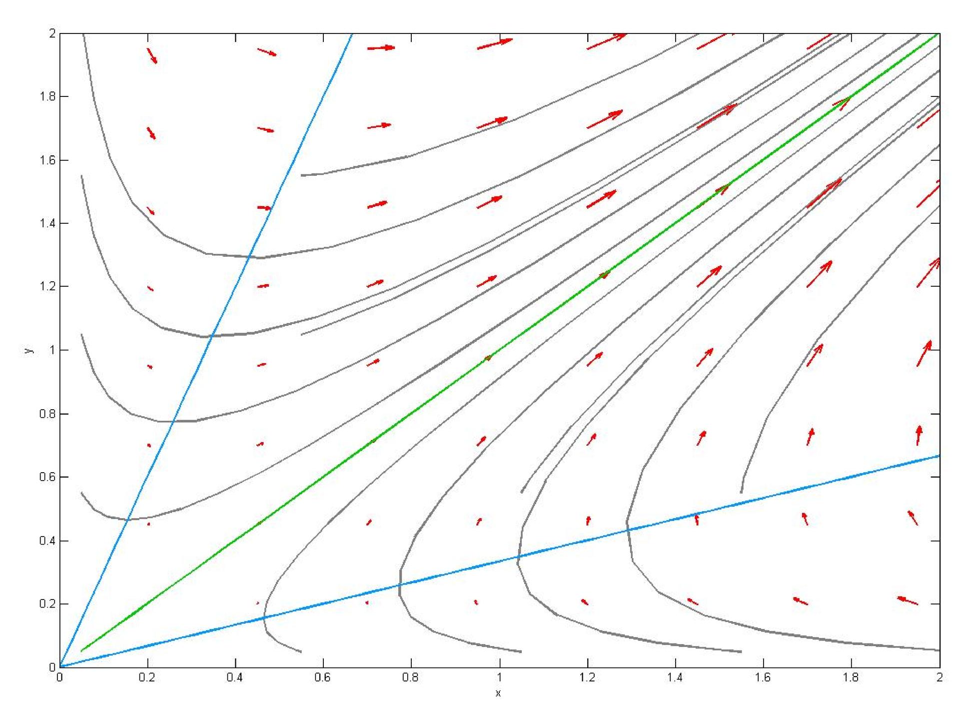

Figure 2: In this case, $a < 0$, $b > 0$ and $\frac{b}{a} > \frac{a}{b}$. Solutions do not collapse and instead blow up in the limit.

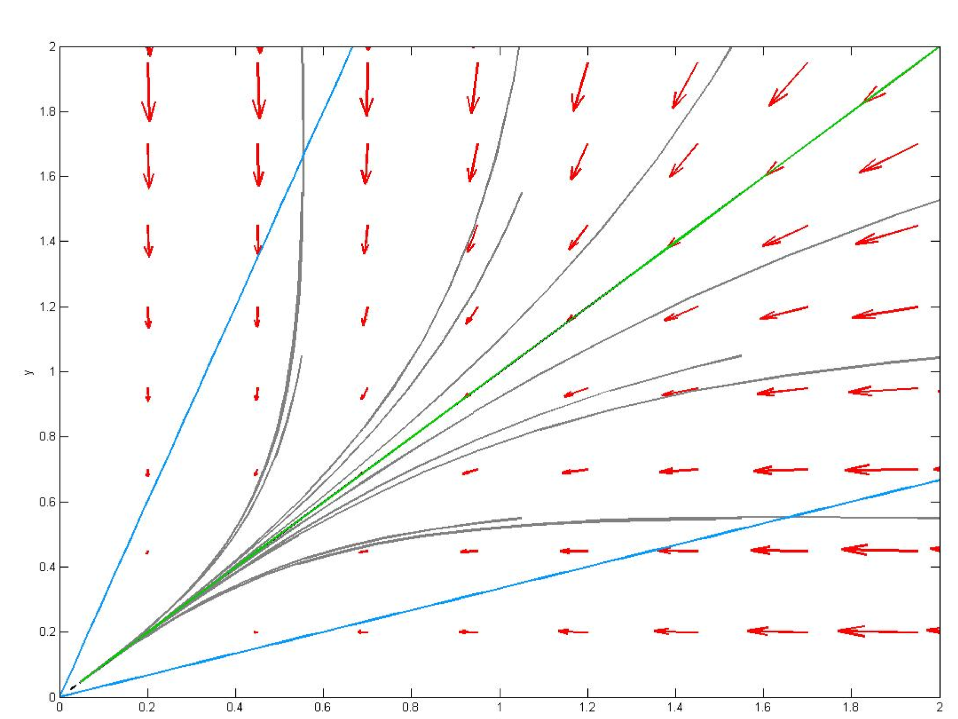

Figure 3: In this case, $a < 0$, $b > 0$ and $\frac{a}{b} > \frac{b}{a}$. Solutions are collapsing and approaching the soliton ray.

References

[1] Bennett Chow and Dan Knopf. The Ricci Flow:An Introduction. American Mathematical Society, 2004.

[2] Dave Glickenstein and Tracy L. Payne. Ricci flow on three-dimensional, unimodular Lie algebras.

[3] James Isenberg, Martin Jackson, and Peng Lu. Ricci flow on locally closed homogeneous 4-manifolds. *Communications in Analysis and Geometry*,14(2) :345-386, 2006.

Support

MK and MR are supported by NSF Grant 0602173. Special thanks are due to Tracy Payne and Mike Williams who advised this project.