CONSTANT SCALAR CURVATURE METRICS ON BOUNDARY COMPLEXES OF CYCLIC POLYTOPES

ANDREW MARCHESE AND JACOB MILLER

ABSTRACT. In this paper we give examples of constant scalar curvature metrics on piecewise-flat triangulated 3-manifolds. These types of metrics are possible candidates for "best" metrics on triangulated 3-manifolds. In the pentachoron, the triangulation formed by the simplicial boundary of the 4-simplex, we find that its stucture is completely deterimed with a vertex transitive metric. Further this metric is a constant scalar curvature metric. Looking at a type of triangulated 3-manifolds, known as boundaries complexes of cyclic polytopes in 4-dimensions, with a metric called a cyclic length metric, we find this entire class of metrics on these manifolds are constant scalar curvature metrics.

1. Notions of Triangulations and Constant Scalar Curvature

It is a classical problem in smooth Reimannian geometry to find constant scalar curvature metric on smooth manifolds. Here we will conisder the same problem in the discrete case. The piecewise-flat triangulated 3-manifold is a discrete version of the compact smooth 3-manifold. These manifolds are formed from 3-dimensional Euclidean tetrahedra, "glued" together in $\mathbb{R}^4$, with various specifications, so that the manifold is without boundary. It is on these manifolds that we will try to find constant scalar curvature metrics. We derive many of these discrete notions of curvature and metrics from [CGY10] and [Gli09].

Since curvature is a geometric concept, before studying curvature one must know the most basic geometric concept of length on these simpilicies. This geometric assignment of lengths is known as a metric on these manifolds.

Definition 1.1. Let (M^3, T) be a triangulation for a 3-manifold and let CM represent the Cayley-Menger determinent. A *metric* ℓ of triangulation (M^3, T) is a complete set of edge lengths for the triangulation such that for each tetrahedron, $t \in \mathrm{T}$, $CM(t) > 0$.

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Definition 1.2. Let (M^3, T) be a triangulated 3-manifold. (M^3, T) is *vertex transitive* if the automorphism group of V, the set of vertices, acts transitively on vertices.

Definition 1.3. A *neighborly triangulation* is one where every vertex forms an edge with every other vertex

Note that a neighborly triangulation is always a vertex transitive triangulation since such a symmetry requires any automorphism of the vertices to act trasitively, since each vertex forms an edge with every other vertex.

These types of triangulations are typically easier to study then others due to their symmetry. In this paper we will only consider neighborly triangulations.

Definition 1.4. Let (M^3, T, ℓ) be a triangulated 3-manifold with metric ℓ. ℓ is a vertex transitive metric if T is *vertex transitive* and if S_v is the set of edge lengths local to v, with multiplicities, then S_v is constant for all v$\in$V.

These will prove to be similarly nice to work because of the geometric symmetries on these metrics.

Remark 1.5. In a neighborly vertex transitive metric, with n vertices, $\sum_{e<v} l_e$ is equal for all v$\in$V. This implies that $\sum_e l_e = \frac{n}{2}\sum_{e<v} l_e$.

We would also like to introduce the notion of a metric within a conformal strucutre as in [CGY10].

Definition 1.6. A *conformal structure* on a triangulated manifold (M^3, T) is a set of metrics determined by a set of real L_{ij} for each edge e_{ij}. Each metric in this structure is then the image under a set of real f_i associated to each vertex v_i under the map:

$$l_{ij} = L_{ij} e^{\frac{1}{2}(f_i+f_j)}$$

Definition 1.7. The set of L_{ij} in the above defintion is called a *conformal class*

These conformal structures allow us to restrict metrics within a conformal class such that edge lengths within a conformal class vary with respect to these f_i.

Once a metric is placed on a triangulation, the manifold has the geometric material necessary to deduce a discrete notion of scalar curvature on piecewise-flat 3-manifolds.

Definition 1.8. Let (M^3, T, ℓ) be a triangulated 3-manifold with metric ℓ. For all e$\in$T, edge curvature, K_e, is defined by:

$$K_e = (2\pi - \sum_t \beta_{e<t}) l_e$$

and for all v∈V, vertex curvature, K_v, is defined by:

$$K_v = \frac{1}{2}\sum_{e>v} K_e$$

with $\beta_{e<t}$ defined as the dihedral angle at edge e inside tetrahedron t.

With these discrete notions of scalar curvature, we also have a discrete analog of the Einstein-Hilbert introduced by Regge known as the Einstein-Hilbert-Regge functional (see [Gli09]).

Definition 1.9. The *Einstein-Hilbert-Regge ($\mathcal{EHR}$)* functional is the sum of the edge curvatures.

$$\mathcal{EHR}(T,\ell) = \sum_e K_e$$

From this we can also derive two very natural functionals called the length-normalized $\mathcal{EHR}$, and the volume normalized $\mathcal{EHR}$.

Definition 1.10. The length-normalized and volume-normalized $\mathcal{EHR}$ ($\mathcal{LEHR}$ and $\mathcal{VEHR}$) functionals are defined to be

$$\mathcal{LEHR}(T,\ell) = \frac{\sum_e K_e}{\sum_e \ell_e}, \quad \mathcal{VEHR}(T,\ell) = \frac{\sum_e K_e}{V^{1/3}}.$$

With these defintions we can deduce two notions of discrete constant scalar curvature metrics, LCSC and VCSC.

Definition 1.11. Let (M^3, T, ℓ) be a triangulated 3-manifold with metric ℓ. ℓ is a LCSC metric if

$$K_v = \lambda_L L_v \tag{1.12}$$

for all v∈V where $\lambda_L = \frac{\sum_e K_e}{\sum_e l_e}$ and $L_v = \frac{1}{2}\sum_{e>v} l_e$

Definition 1.13. Let (M^3, T, ℓ) be a triangulated 3-manifold with metric ℓ. ℓ is a VCSC metric if

$$K_v = \lambda_v V_v \tag{1.14}$$

for all v∈V where $\lambda_v = \frac{\sum_e K_e}{3V}$ and $V_v = \frac{1}{3}\sum_{j,k,l} h_{ijk,l}A_{ijk}$ with V being the total volume of the triangulation. Note that V = $\frac{1}{3}\sum_v V_v$.

As is proven in [CGY10], these LCSC and VCSC metrics occur at the critical points of the $\mathcal{LEHR}$ and $\mathcal{VEHR}$, respectively, with a conformal class.

2. The Pentachoron with a Vertex Transitive Metric

The pentachoron is the smallest simplicial piecewise-flat triangulated 3-manifold, from a combinatoral perspective.

Definition 2.1. The *pentachoron* is the triangulated 3-manifold of S^3 defined by the boundary of the 4-simplex.

This triangulation, with 5 vertices, 10 edges, 10 faces, and 5 tetrahedron, is neighborly. It further reveals a very uniform symmetry with a vertex transitive metric.

Lemma 2.2. *For every vertex transitive metric on the pentachoron, any vertex must have two pairs of edges with equal lengths local to it.*

Proof. Suppose not. Consider a vertex $v_i \in$V. Since the pentachoron is neighborly, v_i is local to four edges, and since their lengths do not occur in pairs, this implies there is a edge local to v_i with length l_1 such that for all other edges e∈E local $v_i, l_e \neq l_1$. Since this is a vertex transitive metric, this is true for all $v_i \in$V. Therefore all edges of length l_1 are local to each vertex exactly once.

Now consider the set S = {e∈ E | $l_e = l_1$}. Then let T={ v∈V | e>v for some e∈S }, where a v∈T is considered to be in T twice if it is local to two distinct edges e∈S. This set is essentially the set of all vertices local to an edge of lenght l_1 with mulitplicities. Further, since all edges are local to exactly two vertices, and T allows for mulitplicities, this means $|T|$ is even. But by the previous observation that all edges of length l_1 are local to each vertex exactly once, this should imply $|T| = |V| = 5$, which is a contradiction. □

Proposition 2.3. *With a vertex transitive metric, every tetrahedron in the pentachoron is isometric to every other tetrahedron in the pentachoron.*

Proof. By Lemma 2.2 there are at most two edge lengths in our metric, say α and β. If $\alpha = \beta$ then the metric is the equal length metric and theorem is trivial. Therefore assume $\alpha \neq \beta$. Choose an arbitrary tetrahedron $t_{ijkl} \in$T. Observe that by Lemma 2.2, vertex v_i has four edges local to it, two with length α and two with length β. From this we will show case-by-case that all the tetrahedra in the pentachoron have the following structure:

Case 1: Let $l_{ij} = l_{ik} = \alpha$ and let $l_{il} = \beta$.
Subcase 1: Let $l_{kl} = \beta$. Therefore, by Prop 1.1, $l_{jl} = \alpha$ and thus $l_{jk} = \beta$ and the tetrahedron is determined.

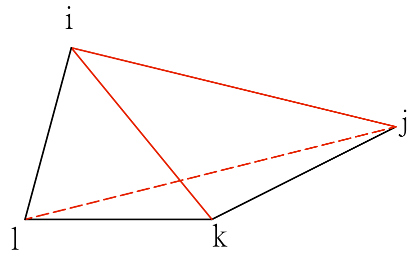

FIGURE 1. Tetrahedron in pentachoron with vertex transitive metric. $l_{ij} = l_{ik} = l_{jl}$ and $l_{il} = l_{jk} = l_{kl}$.

Subcase 2: Let $l_{kl} = \alpha$. Therefore, by Prop 1.1, $l_{jk} = \beta$. Suppose $l_{jl} = \alpha$, and consider the fifth vertex $v_p \in$V. By Prop 1.1 this would imply $l_{lp} = l_{ip} = l_{jp} = \beta$, which is a contradiction to Prop 1.1 since $\beta \neq \alpha$. Therefore $l_{jk} = \beta$, and the tetrahedron is determined. However by relabeling this tetrahedron so that $i' = i, j' = k, k' = j$, and $l' = l$ one can see that then $l_{i'j'} = l_{i'k'} = l_{j'l'} = \alpha$ and $l_{i'l'} = l_{j'k'} = l_{i'l'} = \beta$ which is exactly the tetrahedron in subcase 1.

Case 2: Let $l_{ij} = l_{ik} = \beta$ and let $l_{il} = \alpha$. WLOG we can apply Case 1 and thus $l_{ij} = l_{ik} = l_{jl} = \beta$ and $l_{il} = l_{jk} = l_{il} = \alpha$. But it we rename the vertices such that $i' = k, j' = l, k' = j$, and $l' = i$ one can see then $l_{i'j'} = l_{i'k'} = l_{j'l'} = \alpha$ and $l_{i'l'} = l_{j'k'} = l_{i'l'} = \beta$ which is exactly the tetrahedron in case 1.

Thus for all $t_{ijkl} \in$T we have $l_{ij} = l_{ik} = l_{jl} = \alpha$ and $l_{il} = l_{jk} = l_{il} = \beta$ after at most relabeling, and therefore all tetrahedron in the petachoron with a vertex transitive metric are identical with respect to edge length. □

Due to this isometry between every tetrahedron in the pentachoron with a vertex transitive metric, this type of pentachoron has essentially one unique structure. For example, consider an arbitrary pentachoron with a vertex transitive metric. If the two edge lengths in the metric are α and β, then we can pick an arbitrary tetrahedron and label the vertices such that $l_{ij} = l_{ik} = l_{jl} = \alpha$ and $l_{il} = l_{jk} = l_{kl} = \beta$ as in our standard tetrahedron. If we consider our last vertex v_m, then we have to have $l_{im} = l_{jm} = \beta$ and $l_{km} = l_{lm} = \alpha$. Thus the pentachoron structure is completely determined once we know metric is vertex transitive.

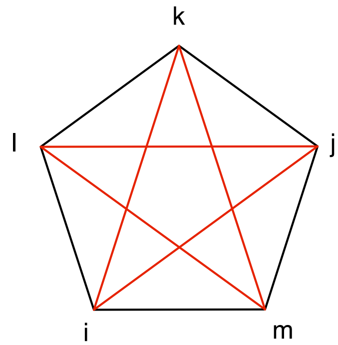

FIGURE 2. 2D Graph of the Vertex Transitive Pentachoron. $l_{im} = l_{mj} = l_{jk} = l_{kl} = l_{li}$ and $l_{ij} = l_{jl} = l_{lm} = l_{mk} = l_{ki}$.

Because of this determined structure on the pentachorn with a vertex transitive metric, we are able to find multiple symmetries in the curvatures and volumes of these pentachorons within all these vertex transitive metrics.

Lemma 2.4. *Consider a pentachoron with a vertex transitive metric. Then K_v is equal for all v$\in V$.*

Proof. Choose an arbitrary tetrahedron t_{ijkl} ∈T. By the proof of Prop 2.3, this tetrahedron's geometric structure is determined. It is now possible to construct the other four tetrahedron (t_{ijkm}, t_{ijlm}, t_{iklm}, t_{jklm}) in T. WLOG, let $l_{ij} = \alpha$. Now note that e_{ij} has exactly three corresponding dihedral angles, one for each tetrahedron it is a part of. In the case of e_{ij}, these are $\beta_{ij,kl}$, $\beta_{ij,km}$ and $\beta_{ij,lm}$. Since $K_e = (2\pi - \sum_t \beta_{e<t})l_e$, $K_e = (2\pi - (\beta_{ij,kl} + \beta_{ij,km} + \beta_{ij,lm})\alpha$ for e_{ij}. But these dihedral angles are the same for all edges with length α, so K_e is equal for all edges with length α. Note that the same proof holds for edges with length β. This implies that $\sum_{e>v} K_e$ is equal for all v∈V since each vertex is local to two edges with length α and two edges with length β. Since $K_v = \frac{1}{2}\sum_{e>v} K_e$, K_v is equal for all v∈V. □

Corollary 2.5. *Under the conditions above, $\sum_e K_e = \frac{5}{2}\sum_{e>v} K_e = 5K_v$ for all v$\in V$.*

Proof. This results follows directly from Lemma 2.4 since we are in a vertex transitive metric. □

These symmetries for the edge curvature also hold for the quantity called V_v in the defintion of the VCSC metric. For a geometric interpretation of what V_v is for any vertex, choose an arbitrary vertex v_i

and a tetrahedron that it is local to, t_{ijkl}. Now find the center point p of tetrahedron t_{ijkl} by constructing the insphere of t_{ijkl} and finding its center. Divide t_{ijkl} into four subtetrahedra, t_{ijkp}, t_{ijlp}, t_{iklp}, and t_{jklp}. Add up the volume of the three subtetrahedra that include v_i. Repeat this construction over all the tetrahedra v_i is local to and sum the results. This sum is V_v for v_i.

Lemma 2.6. *Consider a pentachoron with a vertex transitive metric. For all v∈ V, V_v is constant.*

Proof. Choose an arbitrary tetrahedron t_{ijkl} ∈T. By the proof of Prop 2.3, this tetrahedron's geometric structure is determined. It is now possible to construct the other four tetrahedron (t_{ijkm}, t_{ijlm}, t_{iklm}, t_{jklm}) in T. For any vertex v∈V, $V_{v_i} = \frac{1}{3}\sum_{t>v}\sum_{(j,k,l)<t} h_{ijk,l}A_{ijk}$. Note that each tetrahedron is geometrically identical, so for any tetrahedron, $\sum_{(j,k,l)<t} h_{ijk,l}A_{ijk}$ only varies in regards to which of the four vertices in the tetrahedron we are summing over. But every vertex is part of exactly four tetrahedra, and it acts as a different vertex in each one. In other words, in regards to our standard tetrahedron (see FIGURE 1), every vertex acts as i, j, k, and l exactly once. So for all v∈V, the partial sums of V_v are all the same, and thus V_v is constant for all v∈V. □

From the above symmetries, the following theorem falls out very quickly from the basic computation of LCSC and VCSC.

Theorem 2.7. *The pentachoron with a vertex transitive metric is both LCSC and VCSC.*

Proof. We will begin by showing this metric is LCSC. By Remark 1.5 and Lemma 2.4:

$$\lambda_L = \frac{\sum_e k_e}{\sum_e l_e} = \frac{5K_v}{\frac{5}{2}\sum_{e<v} l_e} = \frac{K_v}{\frac{1}{2}\sum_{e<v} l_e}$$

for all v∈V. Thus, for all v∈V $K_v = \lambda_L\,[\frac{1}{2}\sum_{e<v} l_e]$, therefore the vertex transitive metric is LCSC.

We will now show that this metric is VCSC. By Lemma 2.6, V_v is constant for all v∈V, so $\sum_v V_v = 5V_v$ for all v∈V.
Also, note that the total volume V of the pentachoron is equal to $\frac{1}{3}\sum_v V_v = \frac{5}{3}V_v$ for all v∈V. Thus:

$$\lambda_v = \frac{\sum_e k_e}{3V} = \frac{5K_v}{3V} = \frac{K_v}{V_v}$$

for all v∈V. Thus, for all v∈V, $K_v = \lambda_v V_v$. Therefore, the vertex transitive metric on the pentachoron is VCSC. □

3. Cyclic Length Metrics on Boundaries of 4-Dimensional Cyclic Polytopes

In order to generalize the above case of the pentachoron, we need to define a piecewise-flat 3-manifold and a metric that reflects the much of the symmetry that was found in the vertex transitive pentachoron. These piecewise-flat 3-manifolds we will use are boundary complexes of 4-dimensional cyclic polytopes. We define cyclic polytopes as in [Gru67].

Definition 3.1. A *4-dimensional cyclic polytope*, $C(n,4)$, is the convex hull of n points in $\mathbb{R}^4$, defined for n≥5, such that:

- For each i, 1≤i≤n, there exists a $t_i \in \mathbb{R}$ such that $t_i < t_j$ for all i<j.
- We define the moment curve in $\mathbb{R}^4$ as M_4:$\mathbb{R} \rightarrow \mathbb{R}^4$ by x(t)=(t,t^2,t^3,t^4).
- For each of these n points, $v_i \in \mathbb{R}^4$, 1≤i≤n, $v_i = x(t_i)$.

For these above cyclic polytope $C(n,4)$ define the set V={x(t_i)∈$\mathbb{R}^4$ | 1≤i≤n}. We will consider the following piecewise-flat 3-manifolds:

Definition 3.2. (M^3, T) will be denoted a $\partial C(n,4)$ manifold with n verticies if (M^3, T) is the piecewise-flat simplicial boundry complex of a cyclic polytope in 4-dimensions, $C(n,4)$.

Note in this definition the set V becomes our set of vertices in our manifold.

Further it has been shown that a $\partial C(n,4)$ manifold is a neighborly combinatorial 3-sphere, S^3(see [KL85]). The following condition on the tetrahedron in 4-dimensional cyclic polytopes is due to Gale (see [Gru67]).

Theorem 3.3 (Gale's Evenness Condition)**.** *Consider a cyclic 4-polytope $C(n,4)$. A set of four points $V_4 \subseteq V$ forms a tetrahedron, determined by the convex hull of V_4, inside the convex hull of $C(n,4)$, if and only if every two points in $V \backslash V_4$ are separated on M_4 by an even number of points of V_4.*

Define a cycle $\mathscr{C}$ as follows:

Note that in these defintions M_4 goes through all vertices exactly once. Consider a piecewise-flat version of M_4 formed by all the edges

e_{ij} such that if v_i=x(t_i)∈V, then v_j=x(t_{i+1})∈V. We can add to this path the edge formed by the two vertices v_1=x(t_1) and v_n=x(t_n), which exists since all $\partial C(n,4)$ manifolds are neighborly. Note when we do this these edges form a cycle that goes through all vertices exactly once. We will denote this cycle $\mathscr{C}$.

Corollary 3.4. *A set of four points in a $\partial C(n,4)$ manifold forms a tetrahedron if and only if those four points form two distinct non-local edges in $\mathscr{C}$.*

Proof. Take two distinct non local edges in $\mathscr{C}$. Since these are non-local they form four distinct vertices. Since these four points are on $\mathscr{C}$ they are thus also on M_4 and since they form an edge, they therefore occur in pairs on M_4 or at the end points of M_4, if the edge is formed by v_1 and v_n. Then consider any two points in V different from these four points. These points must be separated by 0, 2, or 4 of our original four points, since they separated by pairs of these points on M_4. Therefore these four points satisfy Gale's Evenness Condition and form a tetrahedron.

Take four vertices in a $\partial C(n,4)$ manifold that form a tetrahedron, and thus satisfy Gale's Evenness Condition. Call this set of vertices V_4. All points in V_4 must form edges on $\mathscr{C}$, else there exists a point not in an edge on $\mathscr{C}$. If this point is v_1 or v_n, then v_2 and v_n or v_1 and v_{n-1} are separated on M_4 by three points in V_4, and this contradicts Gale's Evenness Theorem. If this point is some v_i, i≠1 or n, then v_{i-1} and v_{i+1} are not in V_4 and separated by 1 point on M_4, v_i, in V_4, which contradicts Gale's Evenness Theorem. Therefore all points in V_4 form edges on $\mathscr{C}$. If they only form two edges on $\mathscr{C}$, with all four points being in an edge, then these two edges are clearly distinct and non-local and form the tetrahedron. If these points form three edges on $\mathscr{C}$ then these three edges are distinct, and since n≥5 at least two of these edges are non-local. Then these two edges form our tetrahedron. □

Definition 3.5. Let (M^3, T, ℓ) be a $\partial C(n,4)$ manifold with n vertices. ℓ is called a cyclic length metric if the length of any edge e_{ij} is determined by the minimum number of edges between v_i and v_j on $\mathscr{C}$.

Because this minimum distance is important to defining the cyclic length metric, we will denote the minimum distance between v_i, v_j ∈V on $\mathscr{C}$ as D_{ij}.

Note also that although the lenghts of each edge e_{ij} are determined by D_{ij}, the lengths associated to each distance D on $\mathscr{C}$ must be defined such that the Cayley-Menger Determinent is positive for all tetrahedron, as is specified in the definition a metric.

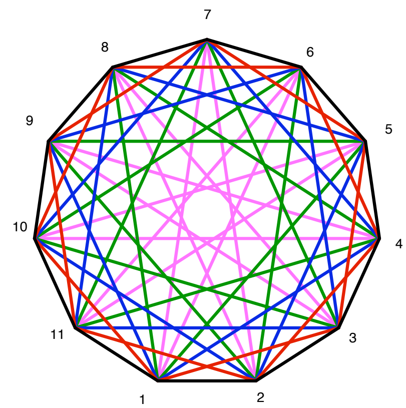

FIGURE 3. A 2-Dimensional Graph Representation of a ∂C(11,4) with a Cyclic Length Metric

As mentioned above LCSC and VCSC metrics occur at critical points of the LEHR and VEHR functionals within a conformal class. Since we will be showing that these cyclic length metrics on $\partial C(n, 4)$ manifolds are LCSC and VCSC we would like to know when these cyclic length metrics occur in a conformal class. However as the next proposition shows, they do not appear in every conformal class on a given $\partial C(n, 4)$ manifold.

Proposition 3.6. *In a $\partial C(n, 4)$ manifold, there exists a conformal class that does not admit a cyclic length metric.*

Proof. Suppose not, then all conformal classes admit a cyclic length metric.

Pick a $v_0 \in$V and label all vertices v_1, v_2,...v_{n-1} such that each v_{i+1} is a step clockwise from v_i on $\mathscr{C}$. Since $n \geq 5$ we know in a cyclic length metric $l_{01} = l_{12} = l_{23} = l_{34}$ and $l_{02} = l_{13} = l_{24}$.

Now consider the conformal class such L_{ij}=1 for all $e_{ij} \in$E, except for L_{12}=$a \neq 1$.

Since $l_{ij} = L_{ij} e^{\frac{1}{2}(f_i + f_j)} = L_{ij} e^{\frac{1}{2} f_i} e^{\frac{1}{2} f_j}$, if the above conformal class admits a cyclic length metric then $l_{01} = l_{12} = l_{23}$.

$\Rightarrow e^{\frac{1}{2}f_0} e^{\frac{1}{2}f_1} = a e^{\frac{1}{2}f_1} e^{\frac{1}{2}f_2} = e^{\frac{1}{2}f_2} e^{\frac{1}{2}f_3}$

$\Rightarrow e^{\frac{1}{2}f_0} = a e^{\frac{1}{2}f_2}$ and $e^{\frac{1}{2}f_3} = a e^{\frac{1}{2}f_1}$

Further $l_{02} = l_{13}$.

$\Rightarrow e^{\frac{1}{2}f_0} e^{\frac{1}{2}f_2} = e^{\frac{1}{2}f_1} e^{\frac{1}{2}f_3}$

$\Rightarrow a(e^{\frac{1}{2}f_2})^2 = a(e^{\frac{1}{2}f_1})^2$

$\Rightarrow e^{f_2} = e^{f_1} \Rightarrow f_2 = f_1$.

Then $l_{12} = l_{34}$.

$\Rightarrow ae^{\frac{1}{2}f_1}e^{\frac{1}{2}f_2} = e^{\frac{1}{2}f_3}e^{\frac{1}{2}f_4}$
$\Rightarrow a(e^{\frac{1}{2}f_2})^2 = ae^{\frac{1}{2}f_1}e^{\frac{1}{2}f_4}$
$\Rightarrow e^{\frac{1}{2}f_1} = e^{\frac{1}{2}f_4}$
But this implies:
$l_{24} = e^{\frac{1}{2}f_2}e^{\frac{1}{2}f_4} = (e^{\frac{1}{2}f_1})^2 = e^{f_1}$
$l_{13} = e^{\frac{1}{2}f_1}e^{\frac{1}{2}f_3} = e^{\frac{1}{2}f_1}(ae^{\frac{1}{2}f_1}) = ae^{f_1}$
and in a cyclic length metric $l_{24} = l_{13}$ which implies:
$ae^{f_1} = e^{f_1} \Rightarrow a = 1$.

This is a contradiction to the definition of a$\neq$1. Therefore this conformal class does not admit a cyclic length metric. □

4. $\partial C(n,4)$ for n-Odd

When considering these cyclic length metrics on $\partial C(n,4)$ manifolds, we require additional information about the combinatorics and symmetries of these manifolds. Due to the slightly different structure of these $C(n,4)$, depending on whether or not n is odd or even, we split these up into two different cases. Here we begin with the odd case.

Lemma 4.1. *Let (M^3, T, ℓ) be a $\partial C(n,4)$ manifold with an odd number of vertices and let ℓ be a cyclic length metric. The triangulation has the following properties:*

1. *There exists an $m \in \mathbb{N}$ such that n = 2m+3.*
2. *There can be up to a total of m+1 distinct edge lengths.*
3. *ℓ is a vertex transitive metric.*
4. *There are 2m^2+3m total tetrahedron in the triangulation.*

Proof. (1) In a $\partial C(n,4)$ manifold n$\geq$5. Since n is odd, n = 2m+1, but since n $\geq$ 5, n = 2m+3 for some m $\in \mathbb{N}$

(2) Choose two arbitrary distinct $v_i, v_j \in$V. Since there are 2m+3 vertices, there are 2m+3 edges in $\mathscr{C}$ by our defintion D_{ij} is the minimum number of edges on $\mathscr{C}$ between v_i nd v_J. Thus there exists a $n_{ij} \in \mathbb{N}$ such that $n_{ij} \geq D_{ij}$ and $n_{ij} + D_{ij} = 2m + 3$.
Therefore, since 2m+3 is odd:
$D_{ij} \neq n_{ij}$
$\Rightarrow n_{ij} > D_{ij}$
$\Rightarrow n_{ij} \geq D_{ij} + 1 > D_{ij}$
$\Rightarrow D_{ij} + D_{ij} + 1 \leq D_{ij} + n_{ij} = 2m + 3$
$\Rightarrow 2D_{ij} + 1 \leq 2m + 3$
$\Rightarrow D_{ij} \leq m + 1$.
Therefore there are only m+1 possibilities for D_{ij}, and since in our metric edge length is a function of D_{ij}, there are at most m+1 possibilities for l_{ij}.

(3) By definition, (M^3, T, ℓ) is a vertex transitive triangulation since it is neighborly. Choose an arbitrary $v_i \in$V. Because all v$\in$V lie on $\mathscr{C}$, and there are 2m+3 edges, we can relabel i as 0, and label the vertices around v_0 by the number of steps clockwise the vertex is in the cycle to v_0. Since there are exactly 2m+3 $= 1 + 2$(m+1) vertices, these vertices can then be labeled as $\{v_{-(m+1)}, v_{-m}, ..., v_0, ...v_{m+1}\}$. Therefore for all v_j, $D_{0j} = |j|$. Thus the set of all D_{0j} with multiplicities is $\{1, 1, ..., m, m, m+1, m+1\}$. Since this is true for all $v_i \in$V, and since lengths of the edges local to v_i are a function of D_{ij} such that v_i and v_j form an edge, this implies that the set of all lengths local to v_i is the same for all $v_i \in$V. Therefore ℓ is a vertex transitive metric.

(4) Note that there are 2m+3 edges in $\mathscr{C}$. Choose an arbitrary edge in $\mathscr{C}$, and a second edge in $\mathscr{C}$ not local to the first edge. By Cor 3.4, all such combinations will completely form all valid tetrahedra. The number of these combinations, and thus total tetrahedra, is exactly:

$$\frac{(2m+3)((2m+3)-3)}{2} = 2m^2 + 3m.$$

□

Note as a consequence of Lemma 4.1, it is clear if there exists a path of distinct p edges between v_i and v_j on $\mathscr{C}$, and p≤m+1 then $D_{ij} = p$. If m+1<p, then D_{ij}=$(2m+3) - p$.

Lemma 4.2. *Let (M^3, T, ℓ) be $\partial C(2m+3,4)$ manifold and let ℓ be a cyclic length metric. There are m distinct types of tetrahedra in the triangulation.*

Proof. By the proof of Lemma 4.1.2, there are at most m+1 edge lengths in the triangulation. Let the set of all these edge lengths, with possible multiplicities, be given by $\{l_1, l_2, ..., l_{m+1}\}$, such that l_{ij}=$l_{D_{ij}}$. Choose two non-local $e_{ij}, e_{kl} \in \mathscr{C}$, and let V_4=$\{v_i, v_j, v_k, v_l\}$.WLOG we can relabel the vertices such that the number of v$\in V_4$ between v_i to v_k on $\mathscr{C}$ is even and $D_{ik} = \min(D_{ik}, D_{jl})$. This can be done for all possible combinations of edges. Then let n=D_{ik}.

Case 1: If $n < m$, then l_{ik}=l_p. Since $e_{ij}, e_{kl} \in \mathscr{C}$, $D_{ij} = D_{kl} = 1$ and $l_{ij} = l_{kl} = l_1$. Since $D_{ik} < D_{jl}$ and an even number of v$\in V_4$ lie between v_i and v_j on $\mathscr{C}$, this implies neither v_j and v_l are on the minimal path on $\mathscr{C}$ between v_i and v_k, else one of the above conditions would be false. Therefore one path on $\mathscr{C}$ inbetween v_i and v_l is the

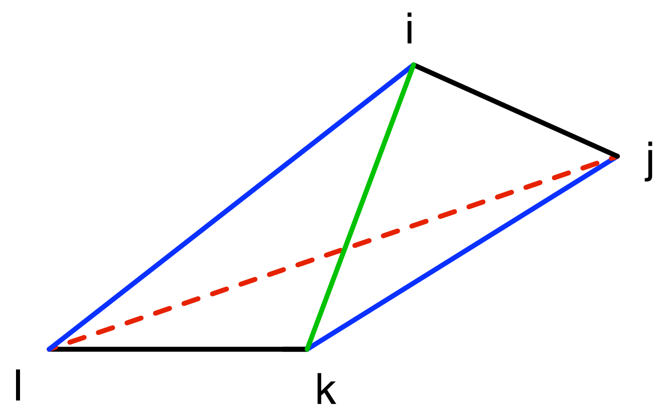

FIGURE 4. Standard n type tetrahedron: $l_{ij} = l_{kl} = l_1$, $l_{ik} = l_n$, $l_{il} = l_{jk} = l_{n+1}$, $l_{jl} = l_{n+2}$

minimal path on $\mathscr{C}$ from v_i to v_k and the edge e_{kl}, which has length n+1 $\leq$ m $<$ m+1. Therefore D_{il} = n+1. Similarly D_{jk} = n+1. Further one path between v_j and v_l on $\mathscr{C}$ is the minimal path of v_i and v_l and the edge e_{ij}, which has length n+2 $\leq$ m+1. Therefore D_{jl}=n+2. Hence $l_{il} = l_{jk} = l_{n+1}$ and $l_{jl} = l_{n+2}$. Therefore the tetrahedron t_{ijkl} is determined with respect to n if n<m.

Case 2: If n=m, then by the same argument as case 1, since m+1 $\leq$ $m+1$, l_{ik}=l_m, l_{ij}=l_{kl}=l_1, and $l_{il} = l_{jk} = l_{m+1}$. However the minimal path between v_j and v_l on $\mathscr{C}$ described in case 1 would have length n+2=m+2. Therefore D_{jl}=m+1, and thus the l_{jl}=l_{m+1} and the tetrahedron in determined.

Case 3: If n $\geq$ m+1, then the length of the path between v_j and v_l formed by the minimal path between v_i and v_k, e_{ij}, and e_{kl}, is greater then m+3. Therefore $D_{jl}$$\leq$m<n. This contradicts our defintion of n= min(D_{ik},D_{jl}). Therefore this case is not possible.

Therefore, from the above cases, the tetrahedron t_{ijkl} is completely determined if we know p. By Cor 3.4 all possible tetrahedra can be described from a combination of edges labeled in the way described above. Thus, for all t$\in$T, t is a function of n. Since by case 3, n$\leq$m, there are at most m distinct type of tetrahedra in the triangulation. However it

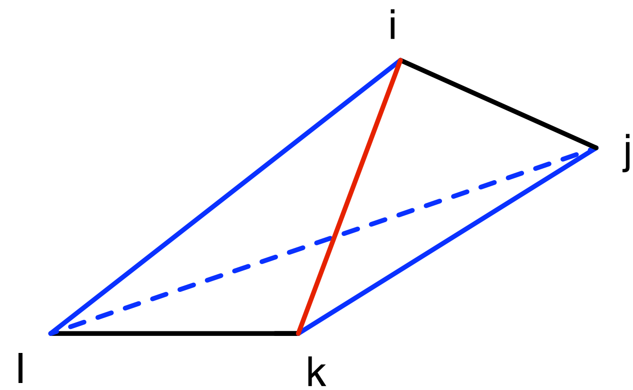

FIGURE 5. Special m type tetrahedron: $l_{ij} = l_{kl} = l_1$, $l_{ik} = l_m$, $l_{il} = l_{jk} = l_{jl} = l_{m+1}$

is clear that we can form a type n tetrahedron for all n, $1 \leq n \leq m$, by taking non-local edges $e_{ij}, e_{kl} \in \mathscr{C}$, with D_{ik}=n, $D_{ik} < D_{il} \leq D_{jl}$. Therefore there are exactly m types of tetrahedra in the triangulation. □

This restriction on the number of various types of tetrahedra for an odd number of vertices allows us to generalize the results found in the vertex transitive pentachoron to the cyclic length metric on our $\partial C(2m + 3, 4)$ manifold.

We will denote these special n type tetrahedra as $t_1, ...t_n, ..., t_m$, where this n is exactly the n denoted in the proof above.

Corollary 4.3. *Let (M^3, T, ℓ) be a $\partial C(2m+3, 4)$ manifold with 2m+3 odd vertices and let ℓ be a cyclic length metric. For each distinct n type tetrahedra, there are 2m+3 tetrahedra of that type in the triangulation.*

Proof. There are exactly 2m+3 edges on $\mathscr{C}$.By Cor 3.4 exactly two tetrahedra of each type t_n, $1 \leq n \leq m$, will be formed with each of these edges. Further, since when forming tetrahedra in this way, edge-by-edge on $\mathscr{C}$, we double count, the total number of t_n, $1 \leq n \leq m$, for each n is $\frac{2(2m+2)}{2} = 2m + 3$. □

Lemma 4.4. *Let (M^3, T, ℓ) be a $\partial C(2m + 3, 4)$ manifold with 2m+3 odd vertices and let ℓ be a cyclic length metric. For all $v \in V$, K_v is constant.*

Proof. First it will be shown that all edges with equal length have equal edge curvature, K_e. We will denote the lengths in this proof similarly as in the proof of Lemma 4.2.Choose an arbitrary e_{ij}∈E with length l_p, such that p=D_{ij}.

Case 1: p≠1, 2, or m+1. Consider the minimal path between v_i and v_j on $\mathscr{C}$. Consider edges e_{ki}, e_{iw}, e_{lj}, and e_{jq} on $\mathscr{C}$, such that v_k and v_l are on this minimal path while v_w and v_q are not. Since p is not 1 or 2 $v_k \neq v_l$. Therefore there are four tetrahedron formed by these four edges, t_{ijkl}, t_{ijwq}, t_{kjiq}, and t_{ilwj}. Note that all other tetrahedra in our triangulation do not contain e_{ij}. By Lemma 4.2, we can determine which n type tetrahedra these are as a function of p. t_{ijwq} is a p type, t_{ijkl} is a p-2 type, and t_{kjiq} and t_{ilwj} are p-1 types. Since in type p and p-2 tetrahedra, there is only one edge with length l_p, this implies $\beta_{ij,kl}$ and $\beta_{ij,wq}$ are determined. Due to the symmetry of the tetrahedra, the two sides of length l_p in a p-1 tetrahedron have the same dehedral angle. This implies $\beta_{ij,kq} = \beta_{ij,lw}$ and are determined. Therefore $\sum_{t>e}\beta_{e,t}$, and thus $K_e = (2\pi - \sum_{t>e}\beta_{e,t})l_e$, is determined by l_p.

Case 2: p=m+1. Lable additional vertices and tetrahedra as in Case 1. Similarly t_{ijkl} is a p-2=m-1 type , t_{kjiq} and t_{ilwj} are p-1=m types, but t_{ijwq} is a m type tetrahedra since $D_{wq} = m$. By the same argument in Case 1, $\beta_{ij,kl}$ is determined. Because of symmetry in the m type tetrahedron, edges of length l_{m+1} local to one edge of length l_{m+1} and one of length l_m have the same dihedral angle. This implies $\beta_{ij,kq} = \beta_{ij,lw}$ and are also determined. In t_{ijwq}, e_{ij} is local to two edges of length l_{m+1} and there is only one edge in a m type tetrahedron such that this occurs, this implies $\beta_{ij,wq}$ is also determined. Therefore $\sum_{t>e}\beta_{e,t}$, and thus $K_e = (2\pi - \sum_{t>e}\beta_{e,t})l_e$, is determined if p=m+1.

Case 3: p=2. This case follows directly from Case 1, however since p=2, $v_k = v_l$. Therefore there is no t_{ijkl} tetrahedron formed, and e_{ij}, is local to only 3 tetrahedra. However in the remaining 3 tetrahedra the proof follows from Case 1.

Case 4: p=1. This means $e_{ij}\in\mathscr{C}$. Therefore e_{ij} is contained in exactly 2m+1 tetrahedra, all choices of edges in $\mathscr{C}$ not local to e_{ij} and the one tetrahedra formed by the two edges local to e_{ij} on $\mathscr{C}$. If we choose an edge e_{kl} on $\mathscr{C}$ not local to e_{ij} such that $D_{ki} = n$ and $D_{kj} = D_{il} = n+1$, then t_{ijkl} is a n type tetrahedron. Similarly we could choose e_{kl} such that $D_{kj} = n$ and $D_{ki} = D_{jl} = n+1$, which is also a n type tetrahedron.

Since we can do this for all n, 1≤n≤m, these form 2m distinct tetrahedra. This, along with the tetrahedron formed by the 2 edges local to e_{ij}, accounts for all 2m+1 tetrahedra with edge e_{ij} contained in it. Therefore e_{ij} is contained in exactly 2 n type tetrahedra for each n>1, and three n=1 type tetrahedra. Due to symmetry in n type tetrahedra, n>1, the dehedral angles are equal for both edges of lenght l_1. Therefore, if we call t_n a n type tetrahedra, then for all n, 2≤n≤m, $\sum_{t_n > e} \beta_{e,t}$ is determined. Because of symmetry in the 1 type tetrahedron, edges of length l_1, local to one edge of length l_1 and one of length l_2, have the same dihedral angle. Further any edge of lenght l_1, local to two edges of length l_1, has a second dihedral angle. In the tetrahedron formed by the two edges local to it, e_{ij} clearly takes on the later roll, and takes on the former role in any other 1 type tetrahedron. Therefore $\sum_{t_1 > e} \beta_{e,t}$ is determined. Finally, since e_{ij} lies in every n type tetrahedra, $\sum_{t>e} \beta_{e,t}$, and thus $K_e = (2\pi - \sum_{t>e} \beta_{e,t}) l_e$, is determined if p=1.

Therefore the edge curvature of any given edge is a function of the length of that edge. Since by Lemma 4.1.3 the metric is vertex transitive, this implies the length of the edges coming into each vertex is the same. These combined imply for all v∈V $\sum_{e>v} K_e$ is constant. Therefore for all v∈V, $K_v = \frac{1}{2} \sum_{e>v} K_e$ is constant. □

Recall the original definition of the $\partial C(n, 4)$, where each vertex was labeled v_i, 1≤i≤n, where $D_{i(i+1)} = 1$. In the following lemmas we will return to labeling vertices in the manner. However we will continue with the convention of having 2m+3 vertices instead of odd n vertices, and we will label n=2m+3 as 0. The motivation for this is clear, due to the the relative location of v_1 and v_{2m+3} in $\mathscr{C}$.

Lemma 4.5. *Any vertex $v_i \in V$ is local to exactly four distinct tetrahedron of type t_k, with 1≤k≤m.*

Proof. WLOG, choose a vertex v_0∈V. It is now possible to construct a tetrahedron type t_k with vertices v_0, v_1, v_{k+1}, v_{k+2}. Let this specific tetrahedron be labeled t_{k1}.

Define a map f:V→V by:
$f(v_i) = v_{i+h(mod 2m+3)}$ where h solves the equation
$0 \equiv k + 1 + h$ (mod 2m+3)
Note that under this map, $v_0 \to v_h$, $v_1 \to v_{h+1}$, $v_{k+1} \to v_0$, and $v_{k+2} \to v_1$. Since this map preserves D_{ij} for any two vertices v_i and v_j, this is still a type t_k tetrahedron. Also note that this tetrahedron is distinct from t_{k1}, since if it was equivelant to it, then:

$h \equiv k+1$ (mod 2m+3)
$\Rightarrow 0 \equiv k+1-h$ (mod 2m+3)
$\Rightarrow k+1+h \equiv k+1-h$ (mod 2m+3)
$\Rightarrow 0 \equiv 2h$ (mod 2m+3) which is a contradiction. Let this tetrahedron be labeled t_{k2}.

Define tetrahedron t_{k3} by vertices $v_{2m+2}, v_0, v_k, v_{k+1}$. This is distinct from t_{k1} and t_{k2} and is clearly a tetrahedron of type t_k.
Now apply f to t_{k3}. Under this map, $v_{2m+2} \rightarrow v_{h-1}$, $v_0 \rightarrow v_h$, $v_{k+1} \rightarrow v_0$, and $v_k \rightarrow v_{2m+2}$. Label this result t_{k4}.

Note that t_{k4} is distinct from t_{k3}, since if it wasn't, then:
$h-1 \equiv k$ (mod 2m+3)
$\Rightarrow h \equiv k+1$ (mod 2m+3)
which is a contradiction by same method as shown above.
Also note that t_{k4} is distinct from t_{k1} since k+1 $\neq$ k+2, and is distinct from t_{k2} since h $\neq$ h+1. Since we can choose any vertex to be v_0, every vertex is part of at least four distinct tetrahedron type t_k, with 1≤k≤m.

But since each vertex is part of four distinct tetrahedron of type t_k, with 1≤k≤m, each vertex lies in at least 4m tetrahedron. Thus, over all 2m+3 vertices, there are 4m(2m+3) vertices, with multiplicities. This is the same as $(2m^2+3m)4$, which is exactly the amount of total tetrahedron multiplied by the number of vertices in each. Thus, every vertex is part of part of exactly four tetrahedron of type t_k, with 1≤k≤m. □

Lemma 4.6. *For all $v \in V$, V_v is constant.*

Proof. For any vertex $v_i \in$V, $V_{v_i} = \frac{1}{3}\sum_{t>v}\sum_{(j,k,l)<t} h_{ijk,l}A_{ijk}$. Note that this sums over every tetrahedron v_i is part of. By Lemma 4.5, each vertex is part of exactly 4m tetrahedron, 4 of each t_k, 1≤k≤m. Thus we can split up V_{v_i} into m distinct parts, one part for every type of tetrahedron. Note that each tetrahedron t_k is geometrically identical, so for any tetrahedron t_k, $\sum_{(j,k,l)<t} h_{ijk,l}A_{ijk}$ only varies in regards to which of the four vertices in the tetrahedron we are summing over.

Consider tetrahedron of type t_k and a vertex v_0 as described in the proof of Lemma 4.5. There are exactly four vertices that make up this tetrahedron (refer to Figure 2). In t_{k1}, v_0 takes the role of vertex l. In t_{k2}, v_0 takes the role of vertex i. In t_{k3}, v_0 takes the role of vertex k. In t_{k4}, v_0 takes the role of vertex j. So v_0 has taken the role of each of the four possible vertices (Note that though the length structure differs in the case of t_m, this doesn't affect the argument; v_0 still takes the role of each vertex exactly once.). Since the selection of v_0 was arbitrary and any vertex can be selected, for all v_i∈V, the partial sum of V_{v_i} with respect to t_k is equal. And since Lemma 4.5 generalizes to every t_k, 1≤k≤m, for all v_i∈V, every partial sum of V_{v_i} is equal, which means that the total sum of V_{v_i} is equal. Thus, for all v∈V, V_v is constant. □

5. $\partial C(n,4)$ for n-Even

The extension of the n-odd case to the n-even is very similar, and will follow in a similar manner as the last section.

Lemma 5.1. *Let (M^3, T, ℓ) be a $\partial C(n,4)$ manifold with an even number of vertices and let ℓ be a cyclic length metric. The triangulation has the following properties:*

(1) *There exists an $m \in \mathbb{N}$, $m > 1$, such that n = 2m+2.*
(2) *There can be up to a total of m+1 distinct edge lengths.*
(3) *ℓ is a vertex transitive metric.*
(4) *There are $2m^2 + m - 1$ total tetrahedron in the triangulation.*

Proof. (1) In a $\partial C(n,4)$ n≥5. Since n is even, n = 2m, but since n ≥ 5, n = 2m+2 for some m ∈ ℕ, m>1.

(2) Choose two arbitrary distinct v_i, v_j ∈V. Since there are 2m+2 vertices, there are 2m+2 edges in $\mathscr{C}$ and by our defintion D_{ij} is the minimum number of edges on $\mathscr{C}$ between v_i nd v_j. Thus there exists a n_{ij}∈ℕ such that $n_{ij} \geq D_{ij}$ and $n_{ij} + D_{ij} = 2m+2$. Therefore:
$\Rightarrow D_{ij} + D_{ij} \leq D_{ij} + n_{ij} = 2m + 2$
$\Rightarrow 2D_{ij} \leq 2m + 2$
$\Rightarrow D_{ij} \leq m + 1.$
Therefore there are only m+1 possibilities for D_{ij}, and since in our metric edge length is a function of D_{ij}, thus there are at most m+1 possibilities for l_{ij}.

(3) By definition, (M^3, T, ℓ) is a vertex transitive triangulation since it is neighborly. Choose an arbitrary v_i∈V. Because all v∈V lie on $\mathscr{C}$, and there are 2m+2 edges, we can relabel i as 0, and label the vertices around v_0 by the number of steps clockwise the vertex is in the cycle to v_0. Since there are exactly 2m+2 = 2(m+1) vertices, these vertices can then be labeled as $\{v_{-(m+1)}, v_{-m}, ..., v_0, ...v_{m+1}\}$. Therefore for all v_j, $D_{0j} = |j|$. Thus the set of all D_{0j} with multiplicities is $\{1, 1, ..., m, m, m+1\}$. Since this is true for all v_i∈V, and since lengths of the edges local to v_i are a function of D_{ij} such that v_i and v_j form an edge,

this implies that the set of all lengths local to v_i is the same for all v_i∈V. Therefore ℓ is a vertex transitive metric.

(4) Note that there are 2m+2 edges in $\mathscr{C}$. Choose an arbitrary edge in $\mathscr{C}$, and a second edge in $\mathscr{C}$ not local to the first edge. By Cor 3.4, all such combinations will completely form all valid tetrahedra. The number of these combinations, and thus total tetrahedra, is exactly:

$$\frac{(2m+2)((2m+2)-3)}{2} = 2m^2 + m - 1.$$

□

Note as a consequence of Lemma 5.1, it is clear if there exists a path of distinct p edges between v_i and v_j on $\mathscr{C}$, and p≤m+1 then $D_{ij} = p$. If m+1<p then D_{ij}=(2m+2) - p.

Lemma 5.2. *Let (M^3, T, ℓ) be a $\partial C(2m+2,4)$ manifold and let ℓ be a cyclic length metric. There are m distinct types of tetrahedra in the triangulation.*

Proof. By the proof of Lemma 5.1.2, there are at most m+1 edge lengths in the triangulation. Let the set of all these edge lengths, with possible multiplicities, be given by $\{l_1, l_2, ..., l_{m+1}\}$, such that l_{ij}=$l_{D_{ij}}$. Choose two non-local e_{ij}, e_{kl}∈$\mathscr{C}$, and let V_4=$\{v_i, v_j, v_k, v_l\}$.WLOG we can relabel the vertices such that the number of v∈V_4 between v_i to v_k on $\mathscr{C}$ is even and $D_{ik} = \min(D_{ik}, D_{jl})$. This can be done for all possible combinations of edges. Then let n=D_{ik}.

Case 1: If $n < m$, then l_{ik}=l_p. Since e_{ij}, e_{kl}∈$\mathscr{C}$, $D_{ij} = D_{kl} = 1$ and $l_{ij} = l_{kl} = l_1$. Since $D_{ik} < D_{jl}$ and an even number of v∈V_4 lie between v_i and v_j on $\mathscr{C}$, this implies neither v_j and v_l are on the minimal path on $\mathscr{C}$ between v_i and v_k, else one of the above conditions would be false. Therefore one path on $\mathscr{C}$ inbetween v_i and v_l is the minimal path on $\mathscr{C}$ from v_i to v_k and the edge e_{kl}, which has length n+1 ≤ m < m+1. Therefore D_{il} = n+1. Similarly D_{jk} = n+1. Further one path between v_j and v_l on $\mathscr{C}$ is the minimal path of v_i and v_l and the edge e_{ij}, which has length n+2 ≤ m+1. Therefore D_{jl}=n+2. Hence $l_{il} = l_{jk} = l_{n+1}$ and $l_{jl} = l_{n+2}$. Therefore the tetrahedron t_{ijkl} is determined with respect to n if n<m.

Case 2: If n=m, then by the same argument as case 1, since n+1 ≤ $m+1$, l_{ik}=l_n, l_{ij}=l_{kl}=l_1, and $l_{il} = l_{jk} = l_{n+1}$. However the minimal path between v_j and v_l on $\mathscr{C}$ described in case 1 would have length

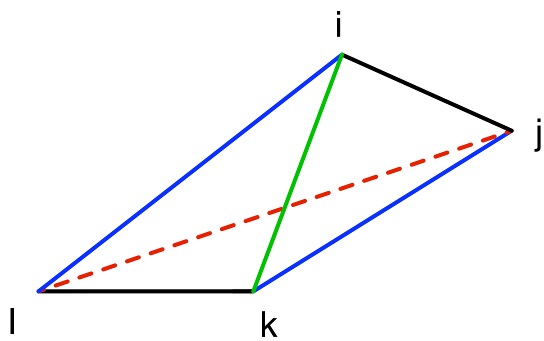

FIGURE 6. Standard n type tetrahedron: $l_{ij} = l_{kl} = l_1$, $l_{ik} = l_n$, $l_{il} = l_{jk} = l_{n+1}$, $l_{jl} = l_{n+2}$

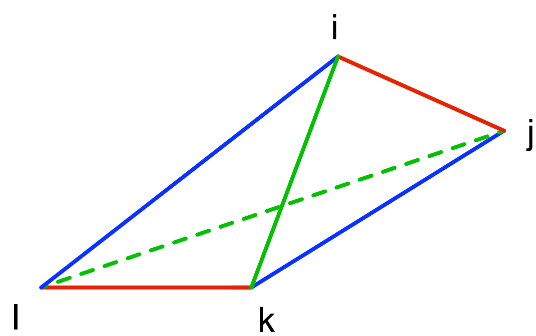

FIGURE 7. Special m type tetrahedron: $l_{ij} = l_{kl} = l_1$, $l_{ik} = l_{jl} = l_m$, $l_{il} = l_{jk} = l_{m+1}$

n+2=m+2. Therefore D_{jl}=m, and thus the l_{jl}=l_m and the tetrahedron in determined.

Case 3: If n $\geq$ m+1, then the length of the path between v_j and v_l formed by the minimal path between v_i and v_k, e_{ij}, and e_{kl}, is greater then m+3. Therefore D_{jl}≤m-1<n. This contradicts our defintion of n= min(D_{ik},D_{jl}). Therefore this case is not possible.

Therefore, from the above cases, the tetrahedron t_{ijkl} is completely determined if we know p. By Cor 3.4 all possible tetrahedra can be described from a combination of edges labeled in the way described above. Thus, for all t∈T, t is a function of n. Since by case 3, n≤m, there are at most m distinct type of tetrahedra in the triangulation. However it is clear that we can form a type n tetrahedron for all n, 1≤n≤m, by taking non-local edges $e_{ij}, e_{kl} \in \mathscr{C}$, with D_{ik}=n, $D_{ik} < D_{il} \leq D_{jl}$. Therefore there are exactly m types of tetrahedra in the triangulation. □

As in the odd case, the restriction on the number of various types of tetrahedra for an even number of vertices allows us to generalize the results found in the vertex transitive pentachoron to the cyclic length metric on our $\partial C(2m+2,4)$ manifold.

We will denote these special n type tetrahedra as $t_1, ...t_n, ..., t_m$, where this n is exactly the n denoted in the proof above.

Corollary 5.3. *Let (M^3, T, ℓ) be a $\partial C(2m+2,4)$ manifold with 2m+2 even vertices and let ℓ be a cyclic length metric. For each distinct n type tetrahedra, n<m-1, there are 2m+2 tetrahedra of that type in the triangulation, and for m type tetrahedra, there are m+1 of them.*

Proof. There are exactly 2m+3 edges on $\mathscr{C}$.By Cor 3.4 exactly two tetrahedra of each type t_n, 1≤n<m, will be formed with each of these edges. Further, each edge will form exactly one t_m type tetrahedra. Since when forming tetrahedra in this way, edge-by-edge on $\mathscr{C}$, we double count, the total number of t_n, 1≤n<m, for each n is $\frac{2(2m+2)}{2} = 2m+3$. Similarly the total number of t_m is $\frac{2m+2}{2} = m+1$. □

Lemma 5.4. *Let (M^3, T, ℓ) be a $\partial C(2m+2,4)$ manifold with 2m+2 even vertices and let ℓ be a cyclic length metric. For all v∈V, K_v is constant.*

Proof. First it will be shown that all edges with equal length have equal edge curvature, K_e. We will denote the lengths in this proof similarly as in the proof of Lemma 5.2.Choose an arbitrary e_{ij}∈E with length l_p, such that p=D_{ij}.

Case 1: p≠1, 2, or m+1. Consider the minimal path between v_i and v_j on $\mathscr{C}$. Consider edges e_{ki}, e_{iw}, e_{lj}, and e_{jq} on $\mathscr{C}$, such that v_k and v_l are on this minimal path while v_w and v_q are not. Since p is not 1 or 2 $v_k \neq v_l$. Therefore there are four tetrahedron formed by these four edges, t_{ijkl}, t_{ijwq}, t_{kjiq}, and t_{ilwj}. Note that all other tetrahedra in our triangulation do not contain e_{ij}. By Lemma 5.2, we can determine which n type tetrahedra these are as a function of p. t_{ijwq} is a p type, t_{ijkl} is a p-2 type, and t_{kjiq} and t_{ilwj} are p-1 types. If p≠m, in type p and p-2 tetrahdra, there is only one edge with length l_p. This implies $\beta_{ij,kl}$ and $\beta_{ij,wq}$ are determined. If p=m there are two possible locations of l_p in p type tetrahedron, but do due to symmetry in the tetrahedron the dihedral angles at both egdes are the same, so $\beta_{ij,wq}$ is still determined. Due to the symmetry of the tetrahedra, the two sides of length l_p in a p-1 tetrahedron have the same dehedral angle also. This implies $\beta_{ij,kq} = \beta_{ij,lw}$ and are determined. Therefore $\sum_{t>e} \beta_{e,t}$, and thus $K_e = (2\pi - \sum_{t>e} \beta_{e,t})l_e$, is determined by l_p.

Case 2: p=m+1. Lable additional vertices and tetrahedra as in Case 1. Similarly t_{ijkl} is a p-2=m-1 type , t_{kjiq} and t_{ilwj} are p-1=m types, but t_{ijwq} is a m-1 type tetrahedra since $D_{wq} = m - 1$. By the same argument in Case 1, $\beta_{ij,kl}$ and $\beta_{ij,wq}$ are determined. Because of symmetry in the m type tetrahedron, once again $\beta_{ij,kq} = \beta_{ij,lw}$ and are also determined. Therefore $\sum_{t>e} \beta_{e,t}$, and thus $K_e = (2\pi - \sum_{t>e} \beta_{e,t})l_e$, is determined if p=m+1.

Case 3: p=2. This case follows directly from Case 1, however since p=2, $v_k = v_l$. Therefore there is no t_{ijkl} tetrahedron formed, and e_{ij}, is local to only 3 tetrahedra. However in the remaining 3 tetrahedra the proof follows from Case 1.

Case 4: p=1. This means $e_{ij} \in \mathscr{C}$. Therefore e_{ij} is contained in exactly 2m tetrahedra, all choices of edges in $\mathscr{C}$ not local to e_{ij} and the one tetrahedra formed by the two edges local to e_{ij} on $\mathscr{C}$. If we choose an edge e_{kl} on $\mathscr{C}$ not local to e_{ij} such that $D_{ki} = n$ and $D_{kj} = n + 1$, then t_{ijkl} is a n type tetrahedron. Similarly we could choose e_{kl} such that $D_{kj} = n$ and $D_{ki} = n + 1$, which is also a n type tetrahedron. Since we can do this for all n, 1≤n≤m, these form 2m-1 distinct tetrahedra, since the same m type tetrahedron is formed by both methods. This, along with the tetrahedron formed by the 2 edges local to e_{ij}, accounts for all 2m tetrahedra with edge e_{ij} contained in it. Therefore e_{ij} is contained in exactly 2 n type tetrahedra for each 2≤n≤m, and three n=1 type tetrahedra, and one n=m type tetrahedron. Due to symmetry in n type tetrahedra, n>1, the dehedral angles are equal for both edges of lenght l_1. Therefore, if we call t_n a n type tetrahedra, then for all n, 2≤n≤m, $\sum_{t_n>e} \beta_{e,t}$ is determined. Because of symmetry in the 1 type tetrahedron, edges of length l_1, local to one edge of length l_1 and one of length l_2, have the same dihedral angle. Further any edge of lenght l_1, local to two edges of length l_1, has a second dihedral angle. In the tetrahedron formed by the two edges local to it, e_{ij} clearly takes on the later roll, and takes on the former role in any other 1 type tetrahedron. Therefore $\sum_{t_1>e} \beta_{e,t}$ is determined. Finally, since e_{ij} lies in every n type tetrahedra, $\sum_{t>e} \beta_{e,t}$, and thus$K_e = (2\pi - \sum_{t>e} \beta_{e,t})l_e$, is determined if p=1.

Therefore the edge curvature of any given edge is a function of the length of that edge. Since by Lemma 4.1.3 the metric is vertex transitive, this implies the length of the edges coming into each vertex is the same. These combined imply for all v∈V $\sum_{e>v} K_e$ is constant. Therefore for all v∈V, $K_v = \frac{1}{2}\sum_{e>v} K_e$ is constant. □

Just as in the odd case, we will recall the original definition of the $\partial C(n,4)$, where each vertex was labeled v_i, 1≤i≤n, where $D_{i(i+1)} = 1$. In the following lemmas we will once again label the vertices $v_0, ..., v_i, ...v_{2m+1}$, for all 0≤i≤2m+1.

Lemma 5.5. *Any vertex v_i∈V is local to exactly four distinct tetrahedron of type t_k, with 1≤k<m, and is local to tetrahedron type t_m exactly twice.*

Proof. WLOG, choose a vertex v_0∈V. It is now possible to construct a tetrahedron type t_k with vertices v_0, v_1, v_{k+1}, v_{k+2}. Let this specific tetrahedron be labeled t_{k1}.

Define a map f:V→V by:
$f(v_i) = v_{i+h(mod 2m+2)}$ where h solves the equation
$0 \equiv k+1+h$ (mod 2m+2)
Note that under this map, $v_0 \to v_h$, $v_1 \to v_{h+1}$, $v_{k+1} \to v_0$, and $v_{k+2} \to v_1$. Since this map preserves D_{ij} for any two vertices v_i and v_j, this is still a type t_k tetrahedron. Let this tetrahedron be labeled t_{k2}. Also note that this tetrahedron is distinct from t_{k1} as long as k<m, since if it was equivelant to it, then:
$h \equiv k+1$ (mod 2m+2)
$\Rightarrow 0 \equiv k+1-h$ (mod 2m+2)
$\Rightarrow k+1+h \equiv k+1-h$ (mod 2m+2)
$\Rightarrow 0 \equiv 2h$ (mod 2m+2) which is a contradiction, as long as $2(k+1) \neq 2(m+1)$, or in other words as long as k≠m. If k=m, then t_1=t_2.

Define tetrahedron t_{k3} by vertices $v_{2m+1}, v_0, v_k, v_{k+1}$. This is distinct from t_{k1} and t_{k2} and is clearly a tetrahedron of type t_k.
Now apply f to t_{k3}. Under this map, $v_{2m+1} \to v_{h-1}$, $v_0 \to v_h$, $v_{k+1} \to v_0$, and $v_k \to v_{2m+1}$. Label this result t_{k4}.

Note that t_{k4} is distinct from t_{k3} as long as k<m, since if it wasn't, then:
$h-1 \equiv k$ (mod 2m+2)

$\Rightarrow h \equiv k+1$ (mod 2m+2)
which is a contradiction by same method as shown above. By the same method as above, if k=m, then t_4=t_3.
Also note that t_{k4} is distinct from t_{k1} since k+1 $\neq$ k+2, and is distinct from t_{k2} since h $\neq$ h+1. Since we can choose any vertex to be v_0, every vertex is part of at least four distinct tetrahedron type t_k, with 1≤k<m, and is part of at least two distinct t_m type tetrahedron.

But since each vertex is part of four distinct tetrahedron of type t_k, with 1≤k<m and two tetrahedron of type t_m, each vertex lies in at least 4(m-1)+2=4m-2 tetrahedron. Thus, over all 2m+2 vertices, there are (4m-2)(2m+2)=8m^2+4m-4 vertices, with multiplicities. This is the same as $(2m^2+m-1)4 = 8m^2+4m-4$, which is exactly the amount of total tetrahedron multiplied by the number of vertices in each. Thus, every vertex is part of part of exactly four tetrahedron of type t_k, with 1≤k<m, and is part of exactly two tetrahedron of type t_m. □

Lemma 5.6. *For all v∈V, V_v is constant.*

Proof. For any vertex v_i∈V, $V_{v_i} = \frac{1}{3}\sum_{t>v}\sum_{(j,k,l)<t} h_{ijk,l}A_{ijk}$. Note that this sums over every tetrahedron v_i is part of. By Lemma 5.5, each vertex is part of exactly 4m-2 tetrahedron, four of each t_k, 1≤k<m and two of type t_m. Thus we can split up V_{v_i} into m distinct parts, one part for every type of tetrahedron. Note that each tetrahedron t_k 1≤k<m is geometrically identical, so for any tetrahedron t_k, $\sum_{(j,k,l)<t} h_{ijk,l}A_{ijk}$ only varies in regards to which of the four vertices in the tetrahedron we are summing over.

Consider tetrahedron of type t_k 1≤k<m and a vertex v_0 as described in the proof of Lemma 5.5. There are exactly four vertices that make up this tetrahedron (refer to Figure 2). In t_{k1}, v_0 takes the role of vertex l. In t_{k2}, v_0 takes the role of vertex i. In t_{k3}, v_0 takes the role of vertex k. In t_{k4}, v_0 takes the role of vertex j. So v_0 has taken the role of each of the four possible vertices. Since the selection of v_0 was arbitrary and any vertex can be selected, for all v_i∈V, the partial sum of V_{v_i} with respect to t_k is equal. And since Lemma 5.5 generalizes to every t_k, 1≤k<m, for all v_i∈V, every partial sum of V_{v_i} is equal up to the t_m case.

In the t_m type tetrahedron, every vertex has the same V_v value due to symmetry, so for each vertex this only depends on how many t_m type tetrahedron the vertex is local to. Since each vertex is part of exactly two distinct t_m type tetrahedron, this partial sum is also the same. This means that the total sum of V_{v_i} is equal for all v∈V. Thus, for all v∈V, V_v is constant. □

6. Cyclic Length Metrics as Constant Scalar Curvature Metrics in $\partial C(n,4)$ Manifolds

Using the above symmetries for $\partial C(n,4)$ with cyclic length metrics, we can now show these metrics to be both LCSC and VCSC.

Corollary 6.1. *In a $\partial C(n,4)$ manifold with a cyclic length metric, if n is the number of vertices in the triangulation, $\sum_e K_e = \frac{n}{2}\sum_{e>v} K_e = nK_v$ for all v∈ V.*

Proof. This results follows directly from Lemmas 4.4 and 5.4 since by 4.1.3 and 5.1.3 a $\partial C(n,4)$ with a cyclic length metric is a vertex transitive metric. □

Theorem 6.2. *Let (M^3, T, ℓ) be a $\partial C(n,4)$ manifold with n vertices and let ℓ be a cyclic length metric. This metric is both LCSC and VCSC.*

Proof. We will begin by showing this metric is LCSC. By Remark 1.5 and Cor 6.1:

$$\lambda_L = \frac{\sum_e k_e}{\sum_e l_e} = \frac{nK_v}{\frac{n}{2}\sum_{e<v} l_e} = \frac{K_v}{\frac{1}{2}\sum_{e<v} l_e}$$

for all v∈V. Thus, for all v∈V $K_v = \lambda_L\ [\frac{1}{2}\sum_{e<v} l_e]$, therefore a $\partial C(n,4)$ manifold with a cyclic length metric is LCSC.

We will now show that this metric is VCSC. By Lemmas 4.6 and 5.6, V_v is constant for all v∈V, so $\sum_v V_v$ = nV_v for all v∈V.
Also, note that the total volume V of the triangulation under any metric is equal to $\frac{1}{3}\sum_v V_v = \frac{n}{3}V_v$ for all v∈V. Thus:

$$\lambda_v = \frac{\sum_e k_e}{3V} = \frac{nK_v}{3V} = \frac{K_v}{V_v}$$

for all v∈V. Thus, for all v∈V, $K_v = \lambda_v V_v$. Therefore, a $\partial C(n,4)$ manifold with a cyclic length metric is VCSC. □

7. Additional Questions for $\partial C(n,4)$ Manifolds

We have now shown that cyclic lenght metrics on $\partial C(n,4)$ manifolds are all VCSC and LCSC. With this information we are motivated to look at several other properties of $\partial C(n,4)$ manifolds and cyclic length metrics on these manifolds.

We have already shown in Prop 3.6 that every conformal does not admit a cyclic length metric. We also know that LCSC and VCSC metrics, and thus cyclic length metrics, occur at critical points of the $\mathcal{LEHR}$ and $\mathcal{VEHR}$ functionals, respectively, within a conformal class. Therefore it would be on interest to find a condition on when a conformal class admits a cyclic length metric.

Further, because these cyclic length metrics occur at ciritical points of the $\mathcal{LEHR}$ and $\mathcal{VEHR}$ functionals, we would like to know the local behavior of these $\mathcal{LEHR}$ and $\mathcal{VEHR}$ functionals.

In the pentachoron we know that all vertex transitive metrics are cyclic length metrics. We have evidence to show that this is not generally true in $\partial C(n,4)$ manifolds, when $n>5$. It is possible that these remaining vertex transitive metrics on $\partial C(n,4)$ manifolds could be classified. It is further possible that some of these metrics are also constant scalar curvature metrcs.

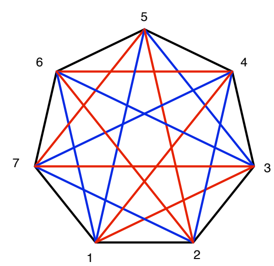

FIGURE 8. A 2D Graph of a $\partial C(7,4)$ with a Vertex Transitive Metric that is not a Cyclic Lenght Metric

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