Background

- Constant Scalar Curvature (CSC) metrics are important to find on piecewise flat triangulations of manifolds because they are possible “best” metrics on these triangulations.
- We use two notions of CSC metrics, Length Constant Scalar Curvature (LCSC) metrics and Vertex Constant Scalar Curvature (VCSC) metrics.

Definition 1: $K_v$, vertex curvature, is the discrete analog of scalar curvature on smooth manifolds.

Definition 2: A metric is called LCSC if:

$$K_v = \lambda L_v$$

for all $v \in V$.

Definition 3: A metric is called VCSC if:

$$K_v = \lambda V_v$$

for all $v \in V$.

Cyclic polytopes

Instead of focusing on all triangulations, consider the triangulations of $S^3$ defined by the boundary of cyclic polytopes.

Definition 4: A cyclic polytope $C(n, 4)$ is a 4-dimensional object defined by $n$ vertices lying on a moment curve. We will be considering $C(n, 4)$ with the moment curve defined by $(t, t^2, t^3, t^4)$.

The boundary of the cyclic polytope $C(n, 4)$ forms a neighborly combinatorial triangulation of $S^3$.

Gale’s evenness condition

A subset of four points $S$ forms a tetrahedron of $C(n, 4)$ if $i < j$ with $i, j \notin S$, then the number of $k \in S$ between $i$ and $j$ is even.

In terms of the 2D graph of the triangulation, this can be interpreted as two outer edges are contained within a tetrahedron if and only if they are not local to each other.

A cyclic length metric

Let $v$ denote the cycle of vertices formed by the moment curve through $v_1, \ldots, v_n$ and the edge $e_{1n}$.

Let $D_{ij}$ denote the smallest number of edges between two vertices $v_i$ and $v_j$ on $v$.

For example, in the boundary of $C(11, 4)$, $D_{13}$ would be 4.

A cyclic length metric is one in which the length of the edge $e_{ij}$ is determined by $D_{ij}$.

The $n$-odd case

Lemma 1: In the triangulation defined by the boundary of $C(n, 4)$ where $n = 2m + 3$ with a cyclic length metric, there are exactly $m$ distinct geometric types of tetrahedron in the triangulation.

Lemma 2: In the triangulation defined by the boundary of $C(n, 4)$ where $n = 2m + 3$ with a cyclic length metric, there are exactly $2m + 3$ type $t_k$ tetrahedron, $1 \leq k \leq m$.

Though there are $m$ types of tetrahedron in the $n$-odd case, type $t_1$, $\ldots$, $t_{m-1}$ have the structure in Figure 2(a) (with differing lengths) while type $t_m$ has the structure in Figure 2(b).

The $n$-even case

Lemma 3: In the triangulation defined by the boundary of $C(n, 4)$ where $n = 2m + 2$ with a cyclic length metric, there are exactly $m$ distinct geometric types of tetrahedron in the triangulation.

Lemma 4: In the triangulation defined by the boundary of $C(n, 4)$ where $n = 2m + 2$ with a cyclic length metric, there are exactly $2m + 2$ type $t_k$ tetrahedron where $1 \leq k \leq m$ and exactly $m + 1$ type $t_m$ tetrahedron.

Similar to the odd case, type $t_1$, $\ldots$, $t_m$ have the structure in Figure 3(a) (with differing lengths) while type $t_m$ has the equihedral structure in Figure 3(b).

Proof of LCSC and VCSC

It is now possible to combine the structure of the even case and the odd case into lemmata that hold for both of the cases.

Lemma 5: In the triangulation defined by the boundary of $C(n, 4)$ with a cyclic length metric $V_v$, is constant for all $v \in V$.

Lemma 6: In the triangulation defined by the boundary of $C(n, 4)$ with a cyclic length metric, $K_v$ is equal for all $v \in V$.

Lemma 7: In the triangulation defined by the boundary of $C(n, 4)$ with a cyclic length metric:

$$L_v = \frac{1}{2} \sum_{i \sim v} t_i$$

is equal for all $v \in V$.

Theorem 1: The triangulation defined by the boundary of $C(n, 4)$ with a cyclic length metric is LCSC and VCSC.

- The proof of this theorem follows directly from Lemmata 5-7.
- Note that the pentachoron corresponds to $n = 5$.

Future questions

- What is the local behavior of the LEHR functional at these cyclic length metrics?
- Which conformal classes can admit a cyclic length metric?
- Do these triangulations admit different LCSC and VCSC metrics in a conformal class that is known to admit a cyclic length metric?

References


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