Constant Scalar Curvature Metrics on the Boundary of $C(n,4)$

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Background

- Constant Scalar Curvature (CSC) metrics are important to find on piecewise flat triangulations of manifolds because they are possible "best" metrics on these triangulations.
- We use two notions of CSC metrics, Length Constant Scalar Curvature (LCSC) metrics and Vertex Constant Scalar Curvature (VCSC) metrics.

Definition 1: K_v, vertex curvature, is the discrete analog of scalar curvature on smooth manifolds.

Definition 2: A metric is called LCSC if:

$$K_v = \lambda_L L_v$$

for all $v \in V$.

Definition 3: A metric is called VCSC if:

$$K_v = \lambda_V V_v$$

for all $v \in V$.

References

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[4] W. Kühnul and G. Lassman. Neighborly Combinatorial 3-Manifolds with Dihedral Automorphism Group. *Israel Journal of Math.*, 52(1-2):147-166, 1985.

Support

- Supported by the National Sscience Foundation under VIGRE grant No 0602173 to the Department of Mathematics at the University of Arizona.
- Advised by Andrea Young. Assisted by Dan Champion and Alex Henniges.

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Cyclic polytopes

Instead of focusing on all triangulations, consider the triangulations of S^3 defined by the boundary of cyclic polytopes.

Definition 4: A cyclic polytope $C(n,4)$ is a 4-dimensional object defined by n vertices lying on a moment curve.

We will be considering $C(n,4)$ with the moment curve defined by (t, t^2, t^3, t^4).

The boundary of the cyclic polytope $C(n,4)$ forms a neighborly combinatorial triangulation of S^3.

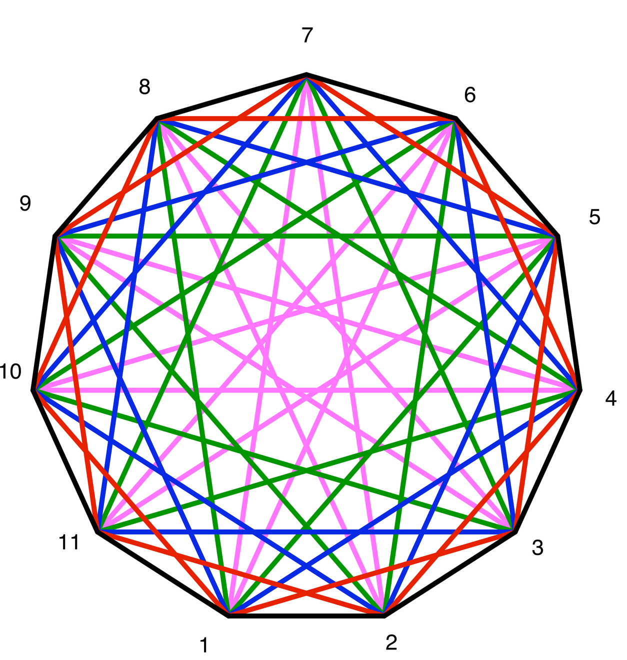

Figure 1: Graph Representation of the boundary of $C(11,4)$.

Gale's evenness condition

A subset of four points S forms a tetrahedron of $C(n,4) \Leftrightarrow$ if $i < j$ with $i, j \notin S$, then the number of $k \in S$ between i and j is even.

- In terms of the 2D graph of the triangulation, this can be interpreted as two outer edges are contained within a tetrahedron if and only if they are not local to each other.

A cyclic length metric

- Let $\mathscr{C}$ denote the cycle of vertices formed by the moment curve through $v_1, \ldots, v_n$ and the edge e_{1n}.
- Let D_{ij} denote the smallest number of edges between two vertices v_i and v_j on $\mathscr{C}$.
  - For example, in the boundary of $C(11,4)$, $D_{1,8}$ would be 4.
- A cyclic length metric is one in which the length of the edge e_{ij} is determined by D_{ij}.

The n-odd case

Lemma 1: In the triangulation defined by the boundary of $C(n,4)$ where $n = 2m+3$ with a cyclic length metric, there are exactly m distinct geometric types of tetrahedron in the triangulation.

Lemma 2: In the triangulation defined by the boundary of $C(n,4)$ where $n = 2m+3$ with a cyclic length metric, there are exactly $2m+3$ type t_k tetrahedron, $1 \leq k \leq m$.

Though there are m types of tetrahedron in the n-odd case, type $t_1, \ldots, t_{m-1}$ have the structure in Figure 2(a) (with differing lengths) while type t_m has the structure in Figure 2(b).

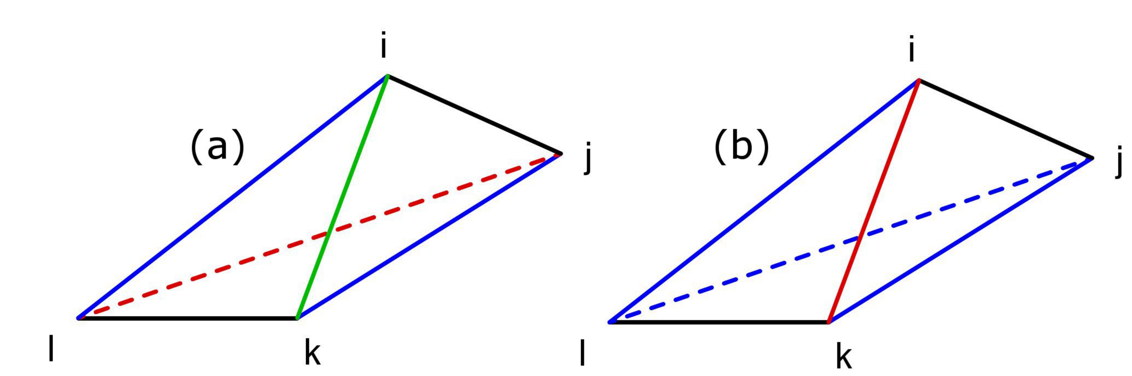

Figure 2: Structure of tetrahedron in $n = 2m+3$ case with a cyclic length metric.

The n-even case

Lemma 3: In the triangulation defined by the boundary of $C(n,4)$ where $n = 2m+2$ with a cyclic length metric, there are exactly m distinct geometric types of tetrahedron in the triangulation.

Lemma 4: In the triangulation defined by the boundary of $C(n,4)$ where $n = 2m+2$ with a cyclic length metric, there are exactly $2m+2$ type t_k tetrahedron where $1 \leq k < m$ and exactly $m+1$ type t_m tetrahedron.

Similar to the odd case, type $t_1, \ldots, t_{m-1}$ have the structure in Figure 3(a) (with differing lengths) while type t_m has the equihedral structure in Figure 3(b).

Proof of LCSC and VCSC

It is now possible to combine the structure of the even case and the odd case into lemmata that hold for both of the cases.

Lemma 5: In the triangulation defined by the boundary of $C(n,4)$ with a cyclic length metric, V_v is constant for all $v \in V$.

Lemma 6: In the triangulation defined by the boundary of $C(n,4)$ with a cyclic length metric, K_v is equal for all $v \in V$.

Lemma 7: In the triangulation defined by the boundary of $C(n,4)$ with a cyclic length metric:

$$L_v = \frac{1}{2} \sum_{e<v} l_e$$

is equal for all $v \in V$.

Theorem 1: The triangulation defined by the boundary of $C(n,4)$ with a cyclic length metric is LCSC and VCSC.

- The proof of this theorom follows directly from Lemmata 5-7.
- Note that the pentachoron corresponds to $n = 5$.

Future questions

- What is the local behavior of the LEHR functional at these cyclic length metrics?
- Which conformal classes can admit a cyclic length metric?
- Do these triangulations admit different LCSC and VCSC metrics in a conformal class that is known to admit a cyclic length metric?

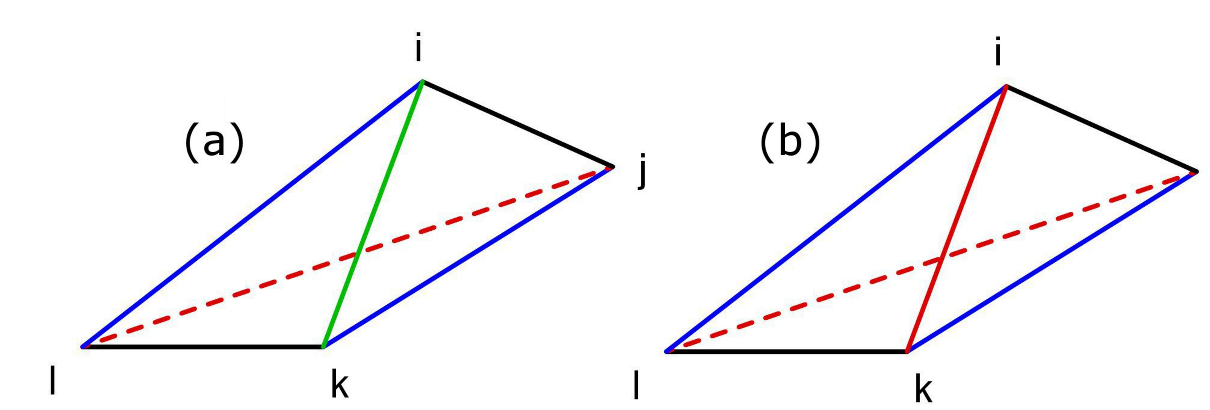

Figure 3: Structure of tetrahedron in $n = 2m+2$ case with a cyclic length metric.