1 Manifolds and triangulations in 2D

Recall that a manifold is a space which is locally homeomorphic to $\mathbb{R}^n$, that is, if $M$ is a manifold, then for every point $x \in M$, there is an open set $U \subset M$ containing $x$ (a “neighborhood of $x$”) such that $U$ is homeomorphic to $\mathbb{R}^n$. Note the following consequence of this definition:

- Note that since $\mathbb{R}^n$ is NOT homeomorphic to $\mathbb{R}^m$ if $m \neq n$ (this is a difficult fact called invariance of domain), every point has a neighborhood homeomorphic to $\mathbb{R}^n$ with the same $n$ for every point. This value of $n$ is called the dimension of $M$. $M$ is then called an $n$-manifold, and often denoted $M^n$.

How do you specify a manifold? We will be interested in a triangulated manifold.

**Definition 1** A triangulation of a 2-manifold is a collection of (abstract) triangles glued together along their boundaries, satisfying certain properties, such that the resulting object satisfies the manifold property.

**Example 2** Not a manifold: three triangles glued along an edge.

**Example 3** Manifolds: boundary of a tetrahedron, icosahedron, octahedron, double triangle, torus, many holed torus

Note that a triangulation has the following pieces as part of it:

- A collection of abstract points called vertices.
- A collection of abstract line segments called edges.
- A collection of abstract triangles called faces. The boundary of the triangle should consist of three (not necessarily distinct) edges.

To be a triangulation, we require the following:

- The ends of an edge should correspond to (not necessarily distinct) vertices.
• The boundary of a face should consist of three (not necessarily distinct) edges.

What makes a triangulation into a manifold? It needs to satisfy the following two properties:

• Each edge neighbors exactly two faces.

• At each vertex, the neighboring faces can be arranged in an order \( f_1, f_2, f_3, \ldots, f_k = f_1 \) such that \( f_j \) and \( f_{j+1} \) intersect at an edge which has the vertex on the boundary.

2 Piecewise flat surfaces

A triangulated manifold is given a piecewise flat structure by specifying lengths of edges in such a way that triangles can be formed from those edge lengths. This suppose the following:

**Proposition 4** Given three positive numbers \( \ell_1, \ell_2, \ell_3 \) such that

\[
\ell_i + \ell_j > \ell_k
\]

for \( \{i, j, k\} = \{1, 2, 3\} \), there is a unique (Euclidean) triangle (up to rigid motion in the plane) with those side lengths.

**Proof.** Place \( \ell_1 \) along the positive x-axis. Attach a line of length \( \ell_2 \) at its second vertex. By rotating the edge, we can get any distance for the third side between \( \ell_1 - \ell_2 \) and \( \ell_1 + \ell_2 \). We have assumed that \( \ell_3 \) is in that range. There are two possible triangles, but they are the same up to rigid motion. 

So, the edge lengths determine the geometry of each of the triangles. Note also that the edge lengths determine angles and areas:

**Proposition 5** The area \( A \) of a triangle can be computed directly from its edge lengths. The angles of a triangle can also be computed from its edge lengths.

**Proof.** The formula for the area of a triangle in terms of edge lengths is called Heron’s formula, and is

\[
A(\ell_1, \ell_2, \ell_3) = \frac{1}{2} \sqrt{(\ell_1 + \ell_2 + \ell_3) (-\ell_1 + \ell_2 + \ell_3) (\ell_1 - \ell_2 + \ell_3) (\ell_1 + \ell_2 - \ell_3)}.
\]

It can also be expressed in terms of the Cayley-Menger determinant:

\[
A = \frac{1}{4} \sqrt{-\det \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & \ell_1 & \ell_2 \\
1 & \ell_1 & 0 & \ell_3 \\
1 & \ell_2 & \ell_3 & 0
\end{pmatrix}}.
\]
Angles can be computed using the law of cosines, which says

\[ \cos \gamma_1 = \frac{\ell_2^2 + \ell_3^2 - \ell_1^2}{2\ell_1 \ell_2} \]

if \( \gamma_1 \) is the angle opposite the side of length \( \ell_1 \).

Finally, we introduce curvature.

**Definition 6** The curvature at the vertex \( v \) of a piecewise flat surface is

\[ K_v = 2\pi - \sum_{f>v} \gamma_{v<f} \]

where the sum is over all faces containing \( v \).

Notice that this is zero precisely if the “flower” of the vertex can be drawn flat in the plane.

It will be useful to have the following invariant of a surface:

**Definition 7** The Euler characteristic of a surface \( M \) is defined to be

\[ \chi(M) = |V| - |E| + |F|, \]

where \(|V|\) is the number of vertices, \(|E|\) is the number of edges, and \(|F|\) is the number of faces.

The important fact is the following:

**Theorem 8** The Euler characteristic depends only on the topology of the manifold, not the triangulation.

So, in particular, any regular polytope has the same Euler characteristic, that of the two-sphere.

The following relates the Euler characteristic to the curvature:

**Theorem 9** (*Discrete Gauss-Bonnet*)

\[ \sum_v K_v = 2\pi \chi. \]

### 3 Three-manifolds

This can be adapted to higher dimensional manifolds, in particular three-dimensions. How do we adapt the surface definition to three-manifolds?

- A triangulation consists of tetrahedra glued together along their boundary.
- To be a manifold, one must have:
  - Every face (triangle) borders exactly 2 tetrahedra.
Around every edge, we can order the tetrahedra bordering it as $t_1, t_2, \ldots, t_k = t_1$ such that $t_j$ and $t_{j+1}$ intersect at a face bordering the edge.

At every vertex the Euler characteristic of the link is 2 (that of the sphere). [The link is the spherical polyhedron determined by the tetrahedra.]

- Volume can be calculated from edge lengths by Cayley-Menger determinant.
- Dihedral angles of a tetrahedron can be computed using spherical law of cosines:
  \[
  \cos \beta_{ij,kl} = \frac{\cos \gamma_{i,kl} - \cos \gamma_{i,jk} \cos \gamma_{i,fl}}{\sin \gamma_{i,jk} \sin \gamma_{i,fl}}
  \]
- Curvature at edges is
  \[
  K_e = \left( 2\pi - \sum_{t>e} \beta_{e<t} \right) \ell_e.
  \]