USING GAP4 IN TEACHING GROUP THEORY

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First Results in an Algebra Course

Let G be a finite group, and let $H < G$ be a subgroup. Then we have that the cosets gH and Hg are all of the same size and they form a partition of G. Hence we get

Lagrange's Theorem:

The order of a subgroup $H < G$ divides the order of G.

A simple but interesting outlook on the further material is then given by the following question:

Question:

To what extent is the converse of Lagrange's theorem true?

That is: Given any divisor $d \mid |G|$, does there exist a subgroup $H < G$ of order $|G|$.

The smallest counterexample to the validity of the converse of Lagrange's theorem is the group A_4 which is of order $|A_4| = 12$ and which has no subgroup of order 6. While it is easily possible to verify this by hand, [1] [4], it is also a chance to use the group program GAP.

So first of all how do we get the symmetric and the alternating group ?

First solution: Use the library of GAP. Define

gap > a4:=AlternatingGroup(4);

gap > Size(a4); [to check the size]

gap > ConjugacyClassesSubgroups(a4);

[to get a list of the conjugacy classes of A_4]

```
gap> a4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> Size(a4);
12
gap> ConjugacyClassesSubgroups(a4);
[ Group( () )^G, Group( [ (1,2)(3,4) ] )^G,
  Group( [ (2,4,3) ] )^G,
  Group( [ (1,3)(2,4), (1,2)(3,4) ] )^G,
  Group( [ (1,3)(2,4), (1,4)(2,3), (2,4,3) ] )^G ]
```

In the screenshot we see that we have obtained something that is not very readable. My opinion is that it is not suitable in this form. So let us change it. First observe that we have 5 objects in our list.

We want to create a list consisting of the size of one representative of each of the 5 conjugacy classes of A_4 by

gap > LSGa5:=ConjugacyClassesSubgroups(a4); [Condensing Notation]

gap > LN:=[ ]; [Creating an empty list]

and then programming a do loop to get the list of subgroup orders:

gap > for k in [1..5] do;

> LN:=Concatenation(LN,[Size(Representative(LSGa5[k]))]);

> od;

and then the command LN will list us the 5 possible group orders:

gap > LN;

$[1, 2, 3, 4, 12]$.

Look at the screenshot for what we have done.

```
gap> a4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> Size(a4);
12
gap> ConjugacyClassesSubgroups(a4);
[ Group( () )^G, Group( [ (1,2)(3,4) ] )^G,
  Group( [ (2,4,3) ] )^G,
  Group( [ (1,3)(2,4), (1,2)(3,4) ] )^G,
  Group( [ (1,3)(2,4), (1,4)(2,3), (2,4,3) ] )^G ]
gap> LSGa5:=ConjugacyClassesSubgroups(a4);
[ Group( () )^G, Group( [ (1,2)(3,4) ] )^G,
  Group( [ (2,4,3) ] )^G,
  Group( [ (1,3)(2,4), (1,2)(3,4) ] )^G,
  Group( [ (1,3)(2,4), (1,4)(2,3), (2,4,3) ] )^G ]
gap> LN:=[];
[  ]
gap> for k in [1..5] do;
> LN:=Concatenation(LN,[Size(Representative(LSGa5[k]))]);
> od;
gap> LN;
[ 1, 2, 3, 4, 12 ]
```

Consider next the group S_4. This does satisfy the converse of Lagrange theorem. Let us verify that in a similar way. But let us another form of input.

gap > s4:=Group((1,2),(1,2,3,4)); [Group Definition by Permutations]

gap > Size(s4); [checking the size]

gap > LSGs4:=ConjugacyClassesSubgroups(s4);

Again we get a list in unreadable form, we can verify that the list has size 11.

Using the same method as before we get the following answer:

```
gap> s4:=Group((1,2),(1,2,3,4));
Group([ (1,2), (1,2,3,4) ])
gap> Size(s4);
24
gap> LSGS4:=ConjugacyClassesSubgroups(s4);
[ Group( () )^G,
  Group( [ (1,4)(2,3) ] )^G,
  Group( [ (1,2) ] )^G,
  Group( [ (2,3,4) ] )^G,
  Group( [ (1,2)(3,4), (1,4)(2,3) ] )^G,
  Group( [ (1,2)(3,4), (1,3,2,4) ] )^G,
  Group( [ (1,2)(3,4), (1,2) ] )^G,
  Group( [ (3,4), (2,3,4) ] )^G,
  Group( [ (1,2)(3,4), (1,3)(2,4), (1,3,2,4) ] )^G,
  Group( [ (1,2)(3,4), (1,3)(2,4), (2,3,4) ] )^G,
  Group( [ (1,2)(3,4), (1,3)(2,4), (2,3,4), (1,3,2,4) ] )^G
]
gap> LNs4:=[];
[  ]
gap> for k in [1..11] do;
>
LNs4:=Concatenation(LNs4,[Size(Representative(LSGS4[k]))]);
> od;
gap> LNs4;
[ 1, 2, 2, 3, 4, 4, 4, 6, 8, 12, 24 ]
```

Now clearly for each of the divisors of S_4 we get at least one conjugacy class of subgroups, but for some divisors we get more than one class. This leads us on to the Sylow theorems.

Bibliography

[1] P.B. Bhattacharya, S.K. Jain, S.R. Nagpaul, Basic Abstract Algebra, Second Edition, Cambridge University Press, 1994

[2] C.R. Jordan, D.A. Jordan, Groups, in Modular Mathematics, Edward Arnold, London 1994

[3] S. MacLane, G. Birkhoff, Algebra, Collier-MacMillan, 1971

[4] Michio Suzuki, Group Theory I, Springer Verlag, Grundlehren 247, New York 1980.