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Modified Ricci flow on a Principal Bundle

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Modified Ricci flow on a Principal Bundle

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Dedicated to my parents who always knew I could do it.

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Modified Ricci flow on a Principal Bundle

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Let M be a Riemannian manifold with metric g, and let P be a principal G-bundle over M having connection one-form a. One can define a modified version of the Ricci flow on P by fixing the size of the fiber. These equations are called the Ricci Yang-Mills flow, due to their coupling of the Ricci flow and the Yang-Mills heat flow. In this thesis, we derive the Ricci Yang-Mills flow and show that solutions exist for a short time and are unique. We study obstructions to the long-time existence of the flow and prove a compactness theorem for pointed solutions. We represent the Ricci Yang-Mills flow as a gradient flow and derive monotonicity formulas that can be used to study breather and soliton solutions. Finally, we use maximal regularity theory and ideas of Simonett concerning the asymptotic behavior of abstract quasilinear parabolic partial differential equations to study the stability of the Ricci Yang-Mills flow in dimension 2 at Einstein Yang-Mills metrics.

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Chapter 1

Introduction

A fundamental problem in differential geometry is finding the "best" metric on a given manifold. In this context "best" is often synonymous with constant curvature. In the last several decades, geometric evolution equations have proved to be highly effective in finding these metrics. The aim of this thesis is to study a natural coupling of two such equations, namely, the Ricci flow and the Yang-Mills heat flow. Roughly speaking, both of these equations are heat equations, thus we expect certain behaviors characteristic of parabolic equations.

In 1982, Richard Hamilton [10] proposed the *Ricci flow* as a means to study 3-manifolds with positive Ricci curvature. Specifically, let (M^n, g) be an n-dimensional Riemannian manifold with metric g. The Ricci flow equations are defined to be

$$\frac{\partial g}{\partial t} = -2Rc, \tag{1.1}$$

where Rc is the Ricci curvature of g. In his seminal paper, Hamilton showed that a closed 3-manifold with positive Ricci curvature is diffeomorphic to a spherical space form, thus that it admits a metric of constant sectional curvature.

Hamilton developed a program intending to use Ricci flow to prove Thurston's geometrization conjecture, which states that every closed manifold admits a geometric decomposition. Hamilton's papers yielded great progress towards this goal, studying a multitude of topics such as singularity formulation [12], compactness theorems [11], and nonsingular solutions [13]. The recent work of Grisha Perelman, which in particular combined comparison geometry and partial differential equations, has provided much progress in the direction of studying geometrization [21], [22]. Additionally, Ricci flow has proven to be a very fruitful area of study in its own right (for overviews, see, e.g., [5], [6]).

On the other hand, the *Yang-Mills heat flow* is a gauge-theoretic heat equation; that is, it is a differential equation for a field on a principal fiber bundle. Let P be a smooth principal G-bundle over a smooth closed manifold M. If A is a connection on P, then A yields an exterior covariant derivative, denoted D_A, that acts on k-forms with values in $\mathcal{G}$, the lie algebra of G. The curvature of A is then $F(A) = D_A A$. We can define the Yang-Mills energy to be

$$\mathcal{YM}(A) = \frac{1}{2}\int_M |F(A)|^2 dV. \tag{1.2}$$

The L^2 gradient flow for this functional is the Yang-Mills heat flow:

$$\frac{\partial A}{\partial t} = -D_A^* F(A), \tag{1.3}$$

where D_A^* is the formal adjoint of D_A. The Yang-Mills heat flow was first used by Atiyah and Bott [1] and by Simon Donaldson [8]. Atiyah and Bott used the

Yang-Mills heat flow to study the topology of minimal Yang Mills connections. Donaldson used it to give an analytic proof of a theorem of Narasimhan and Seshadri concerning the relation between stable holomorphic vector bundles and equivalence classes of Yang-Mills connections. Johan Rade [23] studied the behavior of the Yang-Mills heat flow in two and three dimensions and was able to use a technique of L. Simon to show that solutions converge as $t \to \infty$.

Let P be a $U(1)$-bundle over a compact manifold. One can choose a metric on P such that the Ricci flow equations, with the additional hypothesis that the size of the fiber remains fixed, yield the *Ricci Yang-Mills flow*:

$$\frac{\partial g}{\partial t} = -2Rc + F^2 \tag{1.4a}$$

$$\frac{\partial a}{\partial t} = -d^{\star} F. \tag{1.4b}$$

We would like to use this flow to find the "best" metric on P, and more specifically, we would like to use the bundle curvature to control the flow because the bundle curvature behaves very nicely under this flow and is less complicated than the other curvature tensors. This in part explains the assumption on the size of the fiber; fixing it allows the effect of F to be more pronounced. As we will see below, the difference in signs that terms involving F add to the Ricci tensor of the bundle implies that the bundle metric will certainly not be Einstein. The canonical metric one should hope for is an *Einstein Yang-Mills metric*; namely, one that is Einstein on the base and that has a Yang-Mills connection.

Let us consider a very simple example of Ricci Yang-Mills flow in dimension 2. We can compare it to that of Ricci flow.

Let $M = S^2$ be the 2-sphere, and suppose we have a metric on S^2 of the form

$$g(t) = \Phi(t)\tilde{g},$$

where $\tilde{g}$ is the constant curvature metric having $R = 2$. Now, the Ricci flow equation on S^2 becomes

$$\frac{\partial g}{\partial t} = \frac{d\Phi}{dt}\tilde{g} = -Rg = -\tilde{R}\tilde{g} = -2\tilde{g};$$

i.e. $\frac{d\Phi}{dt} = -2$. Solutions to this equation are of the form $\Phi(t) = \Phi(0) - 2t$, so the solution exists until $T = \frac{\Phi(0)}{2}$ when the manifold shrinks to a round point.

Now let P be a $U(1)$-bundle over S^2 having a a fixed Yang-Mills connection with $F(a) = CdV_{\tilde{g}}$. Here C is some fixed constant. In Lemma 5.3.2, we will explicitly compute the possible values of C. Notice that then $F_i^k F_{kj} = \frac{1}{2}|F|^2 g = \frac{C^2}{2\Phi}\tilde{g}$. Then the Ricci Yang-Mills flow becomes

$$\frac{d\Phi}{dt} = -2 + \frac{C^2}{2\Phi},$$
$$\frac{\partial a}{dt} = 0.$$

Solutions to this equation exist for all time, since $\Phi = \frac{C^2}{4}$ is an asymptotically stable fixed point. Thus we see that including the bundle curvature allows us to avoid the finite time singularity that occurs with Ricci flow. A goal of studying the Ricci Yang-Mills flow would be to find conditions on the bundle curvature such that one could flow past singularities. This example is also instructive in that it demonstrates the lack of scale-invariance in the equation. This will prove to be both a boon and a hindrance in the analysis to come.

We would like to give a brief overview of the contents of this thesis. In the remainder of Chapter 1, we discuss the derivation of the Ricci Yang-Mills flow using a particular metric on a principal bundle. We also compute the evolution equations for several quantities that will be useful later. In Chapter 2, we use the gauge fixing argument of DeTurck to show the short time existence of the Ricci Yang-Mills flow. We also show uniqueness of solutions. We study obstructions to long time existence in Chapter 3 and prove a compactness theorem for the flow, thereby laying the groundwork for singularity analysis. Chapter 4 focuses primarily on monotonic quantities for the Ricci Yang-Mills flow; in particular, we are able to follow ideas of Perelman and show that RYM can be thought of as a gradient flow. We also briefly consider solitons and breathers. Finally, in Chapter 5, we use maximal regularity theory to study the stability of the Ricci Yang-Mills flow at Einstein Yang-Mills metrics.

The Ricci Yang-Mills flow has been studied simultaneously and independently by Jeffrey Streets in [25] and [26]. He has considered the flow in both the two and four dimensional cases.

1.1 Derivation of the Equations

We would like to show how the Ricci Yang-Mills equations can be derived from the Ricci flow equations on a principal bundle. Our specific setting is as follows. Let M be a closed Riemannian manifold with metric $\underline{g}$, and let $U \subset M$ be a local coordinate chart with coordinates $\{x^i\}_{i=1}^n$. Let G be a compact Lie group with smooth metric $\bar{g}$ parametrized by the base. Let $\{y^\theta\}_{\theta=n+1}^m$ be local coordinates on G. Then let $\pi : P \to M$ be a principal G-bundle over M, having connection A. We consider a metric h on the total space P of the form

$$h = \underline{g}_{ij} dx^i dx^j + \bar{g}_{\theta\rho}(dy^\theta + a_k^\theta dx^k)(dy^\rho + a_l^\rho dx^l).$$

Here, $a = \sigma^* A$, where $\sigma : U \to P$ is a smooth local section. We have the following basis for one-forms: $dz^i = dx^i$ and $dz^\theta = dy^\theta + a_i^\theta dx^i$ with the corresponding frame $e_i = \frac{\partial}{\partial x^i} - a_i^\theta \frac{\partial}{\partial y^\theta}$ and $e_\theta = \frac{\partial}{\partial y^\theta}$.

We would like to compute the various curvature quantities in terms of this metric. In the computations to follow, lower case indices will denote quantities on the base, Greek indices will denote quantities on the fiber, and capital indices will indicate both. Additionally underlined quantities are computed with respect to the metric on the base and quantities with a bar are with respect to the fiber metric. F will carry two sets of indices. The Roman ones will correspond to indices on the base. We will always raise or lower the first of these with respect to $\underline{g}$. The Greek index will represent the bundle index; it can be raised or lowered with respect to $\bar{g}$.

In order to derive the Ricci Yang-Mills equations, we would like to make the following two assumptions:

1. $\frac{\partial}{\partial t}\bar{g} = 0$,

2. $\underline{\nabla}\bar{g} = 0$.

As we stated previously, we will see that assumption 1 is important given the difference of sign in the Ricci tensor for terms involving F. Assumption 2 implies that $\bar{g}$ is constant along M. Essentially these requirements amount to keeping the size of the fiber fixed. We will see through the course of the computations that these assumptions result in a natural set of coupled equations.

To begin our computations, we define the structure constants C^P_{MN} of the bundle to be

$$C^P_{MN} e_P = [e_M, e_N].$$

Lemma 1.1.1. *The only non-zero structure constants are of the form* $C^\theta_{ij} = -F^\theta_{ij}$.

Proof. Consider first C^M_{ij}. We have

$$\begin{aligned}
C^M_{ij} e_M &= [e_i, e_j] \\
&= [\frac{\partial}{\partial x^i} - a^\theta_i \frac{\partial}{\partial y^\theta}, \frac{\partial}{\partial x^j} - a^\mu_j \frac{\partial}{\partial y^\mu}] \\
&= [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] - [a^\theta_i \frac{\partial}{\partial y^\theta}, \frac{\partial}{\partial x^j}] - [\frac{\partial}{\partial x^i}, a^\mu_j \frac{\partial}{\partial y^\mu}] + [a^\theta_i \frac{\partial}{\partial y^\theta}, a^\mu_j \frac{\partial}{\partial y^\mu}] \\
&= \frac{\partial}{\partial x^j}(a^\theta_i \frac{\partial}{\partial y^\theta}) - a^\theta_i \frac{\partial}{\partial y^\theta}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^i}(a^\mu_j \frac{\partial}{\partial y^\mu}) + a^\mu_j \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial x^i} \\
&= \frac{\partial}{\partial x^j} a^\theta_i \frac{\partial}{\partial y^\theta} - \frac{\partial}{\partial x^i} a^\mu_j \frac{\partial}{\partial y^\mu} \\
&= -F^\theta_{ij} \frac{\partial}{\partial y^\theta},
\end{aligned}$$

since our structure group is Abelian. In a similar fashion, we compute

$$\begin{aligned} C^M_{i\theta} e_M &= [e_i, e_\theta] \\ &= [\frac{\partial}{\partial x^i} - a^\mu_i \frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\theta}] \\ &= [\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\theta}] - [a^\mu_i \frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\theta}] \\ &= 0. \end{aligned}$$

Finally, we have

$$\begin{aligned} C^M_{\theta\mu} e_M &= [e_\theta, e_\mu] \\ &= [\frac{\partial}{\partial y^\theta}, \frac{\partial}{\partial y^\mu}] \\ &= 0. \end{aligned}$$

□

Lemma 1.1.2. *The Christoffel symbols are given by the following equations:*

$$\begin{aligned} &\Gamma^k_{ij} = \underline{\Gamma}^k_{ij}, &&\Gamma^\alpha_{ij} = -\tfrac{1}{2} F^\alpha_{ij}, \\ &\Gamma^m_{i\theta} = -\tfrac{1}{2} F^k_{\theta i}, &&\Gamma^\alpha_{i\theta} = 0, \\ &\Gamma^k_{\theta\rho} = 0, &&\Gamma^\rho_{\mu\theta} = \bar{\Gamma}^\rho_{\mu\theta}. \end{aligned}$$

Proof. Recall the formula for the Christoffel symbols with respect to a frame:

$$\Gamma^M_{IJ} h_{MK} = \frac{1}{2}(e_I(h_{JK}) + e_J(h_{KI}) - e_K(h_{IJ}) + C^N_{IJ} h_{NK} - C^N_{JK} h_{NI} + C^N_{KI} h_{NJ}). \tag{1.5}$$

We compute as follows, first for $\Gamma^m_{ij} h_{mk}$:

$$\begin{aligned} \Gamma^m_{ij} h_{mk} &= \frac{1}{2}(e_i(h_{jk}) + e_j(h_{ki}) - e_k(h_{ij}) + C^n_{ij} h_{nk} - C^n_{jk} h_{ni} + C^n_{ki} h_{nj}) \\ &= \underline{\Gamma}^m_{ij} \underline{g}_{mk} \end{aligned}$$

Now consider $\Gamma_{ij}^{\alpha} h_{\alpha\rho}$.

$$\begin{aligned}\Gamma_{ij}^{\alpha} h_{\alpha\rho} &= \frac{1}{2}(e_i(h_{j\rho}) + e_j(h_{\rho i}) - e_\rho(h_{ij}) + C_{ij}^{\mu} h_{\mu\rho} - C_{j\rho}^{n} h_{ni} + C_{\rho i}^{n} h_{nj}) \\ &= -\frac{1}{2} F_{ij}^{\mu} \bar{g}_{\mu\rho}.\end{aligned}$$

For $\Gamma_{i\theta}^{m} h_{mk}$, we have

$$\begin{aligned}\Gamma_{i\theta}^{m} h_{mk} &= \frac{1}{2}(e_i(h_{\theta k}) + e_\theta(h_{ki}) - e_k(h_{i\theta}) + C_{i\theta}^{n} h_{nk} - C_{\theta k}^{n} h_{ni} + C_{ki}^{\alpha} h_{\alpha\theta}) \\ &= -\frac{1}{2} F_{ki}^{\alpha} \bar{g}_{\alpha\theta}.\end{aligned}$$

Additionally, we have the computation for $\Gamma_{i\theta}^{\alpha} h_{\alpha\rho}$.

$$\begin{aligned}\Gamma_{i\theta}^{\alpha} h_{\alpha\rho} &= \frac{1}{2}(e_i(h_{\theta\rho}) + e_\theta(h_{\rho i}) - e_\rho(h_{i\theta}) + C_{i\theta}^{\mu} h_{\mu\rho} - C_{\theta\rho}^{n} h_{ni} + C_{\rho i}^{\alpha} h_{\alpha\theta}) \\ &= 0,\end{aligned}$$

since we assume $\underline{\nabla}\bar{g} = 0$. Similarly, $\Gamma_{\theta\rho}^{k} = -\frac{1}{2}\underline{\nabla}^k \bar{g}_{\theta\rho} = 0$. Finally, it is easy to see that $\Gamma_{\mu\theta}^{\rho} = \bar{\Gamma}_{\mu\theta}^{\rho}$. □

Now that we have the formulae for the Christoffel symbols, we can compute the components of the curvature tensor.

Lemma 1.1.3. *The curvature tensor is given by*

$R_{ijk}^{l} = \underline{R}_{ijk}^{l} + \frac{1}{4} F_{ik}^{\alpha} F_{\alpha i}^{l} - \frac{1}{4} F_{ik}^{\alpha} F_{\alpha j}^{l} - \frac{1}{2} F_{ij}^{\alpha} F_{\alpha k}^{l}$,

$R_{ijk}^{\theta} = \frac{1}{2}(\underline{\nabla}_j F_{ik}^{\theta} - \underline{\nabla}_i F_{jk}^{\theta})$,

$R_{\theta j\rho}^{l} = \frac{1}{4} F_{\rho j}^{p} F_{\theta p}^{l}$,

$R^{\beta}_{\theta\mu\rho} = \bar{R}^{\rho}_{\mu\beta\theta}.$

Proof. The components of the curvature tensor can be computed using the formula

$$R^L_{IJK} = e_I(\Gamma^L_{JK}) - e_J(\Gamma^L_{IK}) + \Gamma^M_{JK}\Gamma^L_{IM} - \Gamma^M_{IK}\Gamma^L_{JM} - C^M_{IJ}\Gamma^L_{MK}.$$

Consider first R^l_{ijk}.

$$\begin{aligned} R^l_{ijk} &= e_i(\Gamma^l_{jk}) - e_j(\Gamma^l_{ik}) + \Gamma^m_{jk}\Gamma^l_{im} + \Gamma^\alpha_{jk}\Gamma^l_{i\alpha} - \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^\alpha_{ik}\Gamma^l_{j\alpha} - C^\alpha_{ij}\Gamma^l_{\alpha k} \\ &= \underline{R}^l_{ijk} + \frac{1}{4}F^\alpha_{ik}F^l_{\alpha i} - \frac{1}{4}F^\alpha_{ik}F^l_{\alpha j} - \frac{1}{2}F^\alpha_{ij}F^l_{\alpha k} \end{aligned}$$

Now we compute R^θ_{ijk}.

$$\begin{aligned} R^\theta_{ijk} &= e_i(\Gamma^\theta_{jk}) - e_j(\Gamma^\theta_{ik}) + \Gamma^m_{jk}\Gamma^\theta_{im} + \Gamma^\alpha_{jk}\Gamma^\theta_{i\alpha} - \Gamma^m_{ik}\Gamma^\theta_{jm} - \Gamma^\alpha_{ik}\Gamma^\theta_{j\alpha} - C^\alpha_{ij}\Gamma^\theta_{\alpha k} \\ &= \frac{1}{2}(e_i(-F^\theta_{jk}) - e_j(-F^\theta_{ik}) - \underline{\Gamma}^l_{jk}F^\theta_{il} + \Gamma^l_{ik}F^\theta_{jl}) \\ &= \frac{1}{2}(\underline{\nabla}_j F^\theta_{ik} - \underline{\nabla}_i F^\theta_{jk}). \end{aligned}$$

Then we look at $R^l_{\theta j\rho}$.

$$\begin{aligned} R^l_{\theta j\rho} &= e_\theta(\Gamma^l_{j\rho}) - e_j(\Gamma^l_{\theta\rho}) + \Gamma^\mu_{j\rho}\Gamma^l_{\theta\mu} + \Gamma^k_{j\rho}\Gamma^l_{\theta k} - \Gamma^\mu_{\theta\rho}\Gamma^l_{j\mu} - \Gamma^k_{\theta\rho}\Gamma^l_{jk} - C^M_{\theta j}\Gamma^l_{M\rho} \\ &= \frac{1}{4}F^p_{\rho j}F^l_{\theta p}. \end{aligned}$$

Now we compute $R^\rho_{\mu j\theta}$.

$$\begin{aligned} R^\rho_{\mu j\theta} &= e_\mu(\Gamma^\rho_{j\theta}) - e_j(\Gamma^\rho_{\mu\theta}) + \Gamma^i_{j\theta}\Gamma^\rho_{\mu i} + \Gamma^\alpha_{j\theta}\Gamma^\rho_{\mu\alpha} - \Gamma^i_{\mu\theta}\Gamma^\rho_{ji} - \Gamma^\alpha_{\mu\theta}\Gamma^\rho_{j\alpha} - C^M_{\mu j}\Gamma^\rho_{M\theta} \\ &= 0. \end{aligned}$$

Finally we consider $R^{\beta}_{\theta\mu\rho}$.

$$\begin{aligned} R^{\rho}_{\mu\beta\theta} &= e_{\mu}(\Gamma^{\rho}_{\beta\theta}) - e_{\beta}(\Gamma^{\rho}_{\mu\theta}) + \Gamma^{i}_{\beta\theta}\Gamma^{\rho}_{\mu i} + \Gamma^{\alpha}_{\beta\theta}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{i}_{\mu\theta}\Gamma^{\rho}_{\beta i} - \Gamma^{\alpha}_{\mu\theta}\Gamma^{\rho}_{\beta\alpha} - C^{M}_{\mu\beta}\Gamma^{\rho}_{M\theta} \\ &= \bar{R}^{\rho}_{\mu\beta\theta}. \end{aligned}$$

□

Lemma 1.1.4. *The components of the Ricci tensor have the form*

$$R_{jk} = \underline{R}_{ij} - \tfrac{1}{2}F^{\theta l}_{j}F_{\theta lk},$$

$$R_{j\mu} = -\tfrac{1}{2}\underline{\nabla}^{k}F_{\mu kj},$$

$$R_{\mu\rho} = \tfrac{1}{4}F^{ij}_{\mu}F_{\rho ij}.$$

Proof. We first consider R_{ij}.

$$\begin{aligned} R_{jk} &= g^{MN}R_{MjkN} \\ &= g^{il}R_{ijkl} + \bar{g}^{\theta\rho}R_{\theta jk\rho} \\ &= g^{il}(\underline{R}_{ijkl} + \frac{1}{4}F^{\theta}_{jk}F_{\theta li} - \frac{1}{4}F^{\theta}_{ik}F_{\theta lj} - \frac{1}{2}F^{\theta}_{ij}F_{\theta lk}) + \bar{g}^{\theta\rho}(-\frac{1}{4}F^{p}_{\rho j}F_{\theta kp}) \\ &= \underline{R}_{ij} - \frac{1}{2}F^{\theta l}_{j}F_{\theta lk}. \end{aligned}$$

Next consider $R_{j\mu}$.

$$\begin{aligned} R_{j\mu} &= g^{MN}R_{Mj\mu N} \\ &= g^{ik}R_{ij\mu k} + \bar{g}^{\theta\rho}R_{\theta j\mu\rho} \\ &= g^{ik}(\underline{\nabla}_{i}F_{\mu jk} - \underline{\nabla}_{j}F_{\mu ik}) \\ &= -\frac{1}{2}\underline{\nabla}^{k}F_{\mu kj}. \end{aligned}$$

Finally we have the computation for $R_{\mu\rho}$.

$$\begin{aligned} R_{\mu\rho} &= g^{MN} R_{M\mu\rho N} \\ &= g^{ij} R_{i\mu\rho j} + \bar{g}^{\theta\alpha} R_{\theta\mu\rho\alpha} \\ &= \frac{1}{4} F_{\mu}^{ij} F_{\rho ij}. \end{aligned}$$

□

Lemma 1.1.5. *The scalar curvature is given by* $R = \underline{R} - \frac{1}{4}|F|^2$.

Proof. We have

$$\begin{aligned} R &= g^{MN} R_{MN} \\ &= g^{ij} R_{ij} + \bar{g}^{\mu\rho} R_{\mu\rho} \\ &= \underline{R} - \frac{1}{2} F^{\theta lk} F_{\theta lk} + \frac{1}{4} F^{\rho ij} F_{\rho ij} \\ &= \underline{R} - \frac{1}{4}|F|^2. \end{aligned}$$

□

Let us now consider this metric on a $U(1)$-bundle. In this setting, we can suppress the bundle indices, as the connection and its curvature are actually a 1-form and a 2-form respectively on the base. If we let $\frac{\partial h}{\partial t} = -2Rc$ with our previously stated assumptions, we obtain the following system of equations:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + F_i^k F_{kj}, \tag{1.6a}$$

$$\frac{\partial a_i}{\partial t} = -d^\star F_i. \tag{1.6b}$$

We call these equations the *Ricci Yang-Mills flow* (RYM). Most of our results are concerning Ricci Yang-Mills flow on a surface; i.e. the base manifold will be two-dimensional. Notice that in this case, we can simplify $F_i^k F_{kj}$, which for simplicity, we will denote as Ω. Ω is symmetric, so we can choose an orthonormal basis such that Ω is diagonalized; i.e.

$$\Omega = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

However, we claim that $\lambda_1 = \lambda_2$. In this basis, we have

$$\begin{aligned} \Omega_{11} = F_{11}F_{11} + F_{21}F_{21} &= \lambda_1 \\ \Omega_{12} = F_{11}F_{12} + F_{21}F_{22} &= 0 \\ \Omega_{21} = F_{12}F_{11} + F_{22}F_{21} &= 0 \\ \Omega_{22} = F_{12}F_{12} + F_{22}F_{22} &= \lambda_2. \end{aligned}$$

Since $F_{12} = -F_{21}$, the second or third equation implies that $F_{11} - F_{22} = 0$. However, F is tracefree, so in this basis $F_{11} + F_{22} = 0$. Thus $F_{11} = F_{22} = 0$. Then $F_{21}^2 = \lambda_1$ and $F_{12}^2 = \lambda_2$. Thus, $\lambda_1 = \lambda_2$.

Then we have the following computation in this basis: $|\Omega|^2 = 2\lambda_1^2$ and $|F|^4 = 4\lambda_1^2$. So $|\Omega|^2 - \frac{1}{2}|F|^4 = 0$. Notice that this implies that the tracefree quantity $\dot{\Omega}_{ij} := \Omega - \frac{1}{2}|F|^2 g_{ij} = 0$ because $|\dot{\Omega}|^2 = |\Omega|^2 - |F|^2 tr_g(\Omega) + \frac{1}{2}|F|^4 = |\Omega|^2 - \frac{1}{2}|F|^4$. So in dimension 2, we can instead write the Ricci Yang-Mills equations as

$$\frac{\partial g_{ij}}{\partial t} = -Rg_{ij} + \frac{1}{2}|F|^2 g_{ij}, \tag{1.7a}$$
$$\frac{\partial a_i}{\partial t} = -d^\star F_i. \tag{1.7b}$$

1.2 Evolution Equations

We would like to compute the evolution equations of various geometric quantities that will prove useful in the sequel. For the rest of the paper, we will use equation (1.7) unless otherwise stated. We will also drop the notation $\underline{g}$ for the base metric and choose just to write g. For more detail about standard variation formulas, see Chapter 3 of [5].

Lemma 1.2.1. *The Christoffel symbols* Γ_{ij}^k *evolve by*

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}(\nabla_i \phi \delta_j^k + \nabla_j \phi \delta_i^k - \nabla^k \phi g_{ij}), \tag{1.8}$$

where $\phi = -R + \frac{1}{2}|F|^2$*.*

Proof. We use equation (1.5) to compute the variation of Γ_{ij}^k. Since structure constants of the form C_{ij}^k are zero, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_{ij}^k &= \frac{1}{2}(\partial_t g^{kl})(e_i(g_{jl}) + e_j(g_{il}) - e_l(g_{ij})) + \frac{1}{2}g^{kl}(\partial_t e_i(g_{jl}) + \partial_t e_j(g_{il}) - \partial_t e_l(g_{ij})) \\
&= (-\phi)\Gamma_{ij}^k + \frac{1}{2}g^{kl}(e_i(\partial_t g_{jl}) + e_j(\partial_t g_{il}) - e_l(\partial_t g_{ij})) \\
&= (-\phi)\Gamma_{ij}^k + \frac{1}{2}(\nabla_i(\phi)\delta_j^k + \nabla_j \phi \delta_i^k - \nabla^k \phi g_{ij}) \\
&+ \frac{1}{2}g^{kl}\phi(\Gamma_{ij}^m g_{ml} + \Gamma_{il}^m g_{jm} + \Gamma_{ji}^m g_{ml} + \Gamma_{jl}^m g_{im} - \Gamma_{li}^m g_{mj} - \Gamma_{lj}^m g_{im}),
\end{aligned}$$

where we used $\frac{\partial}{\partial y^\theta} g_{ij} = 0$ in the second equality and the fact that $0 = \nabla_i g_{jk} = e_i(g_{jk}) - \Gamma_{ij}^m g_{mk} - \Gamma_{ik}^m g_{jm}$ in the third. Notice that $\Gamma_{il}^m - \Gamma_{li}^m = C_{il}^m = 0$ and similarly for $\Gamma_{jl}^m - \Gamma_{lj}^m$. We are left with

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_{ij}^k &= (-\phi)\Gamma_{ij}^k + \frac{1}{2}(\nabla_i(\phi)\delta_j^k + \nabla_j \phi \delta_i^k - \nabla^k \phi g_{ij}) + \phi \Gamma_{ij}^k \\
&= \frac{1}{2}(\nabla_i(\phi)\delta_j^k + \nabla_j \phi \delta_i^k - \nabla^k \phi g_{ij}).
\end{aligned}$$

. $\square$

Lemma 1.2.2. *The variation of the scalar curvature is given by*

$$\frac{\partial}{\partial t}R = \Delta R - \frac{1}{2}\Delta|F|^2 + 2R^2 - R|F|^2. \tag{1.9}$$

Proof. We would like to use the standard variation formula $\partial_t R = -\Delta H + \nabla^p\nabla^q h_{pq} - \langle h, Rc\rangle$, where $H = tr_g h$.

$$\begin{aligned}
\partial_t R &= -\Delta(g^{ij}(-Rg_{ij} + \frac{1}{2}|F|^2 g_{ij})) + \nabla^p\nabla^q(-Rg_{pq} + \frac{1}{2}|F|^2 g_{pg}) \\
&- \langle(-Rg_{ij} + \frac{1}{2}|F|^2 g_{ij}), \frac{R}{2}g_{ij}\rangle \\
&= 2\Delta R - \Delta|F|^2 - \Delta R - \Delta R + \frac{1}{2}\Delta|F|^2 + 2R^2 - R|F|^2 \\
&= \Delta R - \frac{1}{2}\Delta|F|^2 + 2R^2 - R|F|^2.
\end{aligned}$$

$\square$

Lemma 1.2.3. *The bundle curvature F evolves by*

$$\frac{\partial}{\partial t}F = \Delta F, \tag{1.10}$$

and $|F|^2$ evolves by

$$\frac{\partial}{\partial t}|F|^2 = \Delta|F|^2 - 2|\nabla F|^2 + 2R|F|^2 - |F|^4. \tag{1.11}$$

Proof. First we consider the evolution of F. Since our structure group is abelian, $F = da$. Thus we can compute

$$\partial_t F = d\partial_t a = -dd^\star F.$$

By the Bianchi identity, $dF = 0$. Thus we can write the evolution equation of F as

$$\partial_t F = -dd^\star F - dd^\star F = \Delta_d F.$$

We claim that on a surface, the Hodge Laplacian on 2-forms is equivalent to the rough Laplacian. Recall the following standard formula, where α is some 2-form:

$$\Delta_d \alpha_{ij} = \Delta \alpha_{ij} + g^{kp} g^{lq} R_{ijkl} \alpha_{pq} - g^{kl} R_{ik} \alpha_{lj} - g^{kl} R_{jk} \alpha_{il}.$$

On a surface, $R_{ijkl} = \frac{R}{2}(g_{il}g_{jk} - g_{ik}g_{jl})$ and $R_{ij} = \frac{R}{2} g_{ij}$. Plugging these identities into the equation above yields the result.

Next we consider the evolution of $|F|^2$ with respect to the evolving metric $g(t)$.

$$\begin{aligned}
\partial_t |F|^2 &= \partial_t (g^{ij} g^{kl} F_{ik} F_{jl}) \\
&= 2(R g^{ij} - \frac{1}{2}|F|^2 g^{ij}) g^{kl} F_{ik} F_{jl} + 2 g^{ij} g^{kl} \Delta_d F_{ik} F_{jl} \\
&= 2 g^{ij} g^{kl} \Delta_d F_{ik} F_{jl} + 2R|F|^2 - |F|^4 \\
&= \Delta |F|^2 - 2|\nabla F|^2 + 2R|F|^2 - |F|^4.
\end{aligned}$$

□

Lemma 1.2.4. *$d^\star F$ obeys the following evolution equation:*

$$\frac{\partial}{\partial t} d^\star F_j = \Delta d^\star F_j + \frac{1}{2}(R - |F|^2) d^\star F_j - \nabla^i (R - \frac{1}{2}|F|^2) F_{ik}. \qquad (1.12)$$

Proof. We would like to use that $d^\star F_j = -g^{ik}\nabla_k F_{ij}$ and to compute the evolution of $g^{ij}\nabla_i F_{jk}$.

$$\begin{aligned}\partial_t g^{ij}\nabla_i F_{jk} &= \partial_t(g^{ij})\nabla^i F_{jk} + \nabla^j \partial_t F_{jk} - g^{ij}(\partial_t(\Gamma^m_{ij}F_{mk} - \Gamma^m_{ik}F_{jm}) \\ &= -\phi\nabla^j F_{jk} + \nabla^j \Delta F_{jk} - \nabla^j(\phi)F_{jk},\end{aligned}$$

where we use equation (1.8) to obtain the final term. All that remains is to commute ∇ and Δ. Using the standard commutator formulas, one can easily show that $\nabla^j\Delta F_{jk} = \Delta\nabla^j F_{jk} - \frac{R}{2}\nabla^j F_{jk}$. The result follows. □

The variation of other geometric quantities can be computed in a similar fashion.

Chapter 2

Short-time Existence and Uniqueness

2.1 Short-time Existence

In 1982 Richard Hamilton proposed the Ricci flow equations for a Riemannian manifold [10]:

$$\frac{\partial g}{\partial t} = -2Rc$$

Due to the diffeomorphism invariance of the Ricci tensor, this is only a weakly parabolic system of equations, so one cannot directly apply parabolic theory to the problem of short-time existence of solutions. Hamilton initially proved that solutions to Ricci flow do exist by using the Nash-Moser implicit function theorem. However, in 1983, Dennis DeTurck showed short-time existence by defining a modified flow that was parabolic and that differed from Ricci flow only by a one parameter family of diffeomorphisms [7].

We would like to use DeTurck's method to prove short-time existence for equations (1.6), which are essentially the Yang-Mills heat equation coupled to the Ricci flow. Our flow is not the full Ricci flow, as we make some simplifying assumptions. Thus short-time existence of our system does not follow directly from that of Ricci flow. We are instead left with a system of two coupled weakly parabolic equations.

We want to define a modified Ricci Yang-Mills flow (GRYM) that is parabolic and such that solutions to GRYM differ from those of RYM only by diffeomorphisms. We will then appeal to parabolic existence theory, as in [7], to get existence of solutions (g, a) to GRYM on some interval $[0, \epsilon)$. By pulling back those solutions, we will obtain the solution to RYM $(\hat{g}, \hat{a})$.

We notice that neither of our equations are parabolic–equation (1.6a) due to diffeomorphism invariance and equation (1.6b) due to gauge invariance. We would like to break this symmetry. So we define the modified flow as DeTurck does: We make the standard choice of vector field, as used in short time existence proofs. Namely, let $W^k = g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k)$, for $k = 1, \ldots, n$, where $\tilde{\Gamma}$ is the Christoffel symbol with respect to a fixed background metric $\tilde{g}$. Additionally, let $\tilde{A}$ be a fixed connection on P. If $A = A(t)$ is a time-dependent connection on P, then the difference $a(t) := A - \tilde{A}$ is a one-form on M. Then we can let $W^{n+1} = -d^\star(A - \tilde{A}) = -d^\star a$. We can compute the lie derivative of h with respect to W as follows. Let $i, j = 1, \ldots, n$. Then

$$\begin{aligned}
\mathcal{L}_W h_{ij} &= \nabla_i W_j + \nabla_j W_i \\
&= \mathcal{L}_W g_{ij} - \Gamma_{ij}^{n+1} W_{n+1} - \Gamma_{ji}^{n+1} W_{n+1} \\
&= \mathcal{L}_W g_{ij} - \frac{1}{2} F_{ij}(-d^\star a) - \frac{1}{2} F_{ji}(-d^\star a) \\
&= \mathcal{L}_W g_{ij}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\mathcal{L}_W h_{in+1} &= \nabla_i W_{n+1} + \nabla_{n+1} W_i \\ &= -\nabla_i d^\star a \\ &= -dd^\star a_i.\end{aligned}$$

Finally, one can see that $\mathcal{L}_W h_{n+1n+1} = 0$. Thus we are motivated to define the modified Ricci Yang-Mills flow (GRYM) to be

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + F_i^k F_{kj} + \mathcal{L}_W g_{ij} \tag{2.1a}$$

$$\frac{\partial a_i}{\partial t} = -d^\star F_i - dd^\star a_i. \tag{2.1b}$$

Both of the equations are parabolic, so standard parabolic existence theory implies that we have solutions (g, a) on some time interval $[0, \epsilon)$. We can also notice that the one parameter family of vector fields $W(t)$ exists as long as solutions do and that they can be used to generate a one parameter family of diffeomorphisms ϕ_t via the equations:

$$\begin{aligned}\frac{\partial}{\partial t}\phi_t(p) &= -W(\phi_t(p), t) \\ \phi_0 &= id_M.\end{aligned}$$

We claim that $\phi_t^* h$, under the same assumptions as before, solves our RYM equations. Using equation (1.6a), we see

$$\begin{aligned}
\frac{\partial}{\partial t}(\phi_t^* g(t)) &= \frac{\partial}{\partial s}\Big|_{s=0} (\phi_{t+s}^* g(t+s)) \\
&= \phi_t^*(\frac{\partial}{\partial t} g(t)) + \frac{\partial}{\partial s}\Big|_{s=0} \phi_{t+s}^* g(t) \\
&= \phi_t^*(-2Rc(g(t)) + \mathcal{L}_{W_t} g(t) + F^2(g(t))) - \mathcal{L}_{(\phi_t^{-1})_* W_t}(\phi_t^* g(t)) \\
&= -2Rc(\phi_t^* g(t)) + \phi_t^* \mathcal{L}_{W_t} g(t) + (F^2(\phi_t^* g(t)) - \phi_t^* \mathcal{L}_{W_t} g(t)) \\
&= -2Rc(\phi_t^* g(t)) + F^2(\phi_t^\star g(t)).
\end{aligned}$$

(c.f. [5]) A similar computation for a yields

$$\begin{aligned}
\frac{\partial}{\partial t}\phi_t^* a_t &= \frac{\partial}{\partial s}\Big|_{s=0} (\phi_{t+s}^* a_{t+s}) \\
&= \phi_t^*(\frac{\partial}{\partial t} a_t) + \frac{\partial}{\partial s}\Big|_{s=0} \phi_{t+s}^* a_t \\
&= \phi_t^*(-d^\star F - dd^\star a) + \frac{\partial}{\partial s}\Big|_{s=0} ((\phi_t^{-1} \circ \phi_{t+s})^* \phi_t^* a) \\
&= \phi_t^*(-d^\star F) + \phi_t^* \mathcal{L}_W a - \mathcal{L}_{(\phi_t^{-1})_* V}(\phi_t^* a) \\
&= -d^\star F(\phi_t^* a).
\end{aligned}$$

So we see that $(\phi_t^* a, \phi_t^* g)$ does indeed solve RYM on $[0, \epsilon)$, and so we have proved the following theorem:

Theorem 2.1.1. *There exists an $\epsilon > 0$ such that the Ricci Yang-Mills equations have a solution (g, a) for t in the interval $[0, \epsilon)$.*

2.2 Uniqueness

We would like to prove the uniqueness of solutions to the Ricci Yang-Mills flow. We will use a technique similar to that in Chapter 3 of [5]; namely, we will show that the DeTurck diffeomorphisms satisfy a certain set of equations that will guarantee uniqueness.

Fix a background metric $\tilde{g}$ on M and a background connection $\tilde{A}$ on P. Suppose that $(\bar{g}_1(t), \bar{a}_1(t))$ and $(\bar{g}_2(t), \bar{a}_2(t))$ are solutions to the Ricci Yang-Mills flow on the same time interval with the same initial data. For $i = 1, 2$, let $(\phi_1^i)_t$ denote the solution of the harmonic map heat flow with respect to $\bar{g}_1$; i.e.

$$\partial_t(\phi_1^i)_t = \Delta_{\bar{g}_1, \tilde{g}}(\phi_1^i)_t. \tag{2.2}$$

Similarly, let $(\phi_2^i)_t$ denote the solution of the harmonic map heat flow with respect to $\bar{g}_2$. In addition, for $j = 1, 2$, let $(\phi_j^3)_t$ denote the solution to

$$\partial_t(\phi_j^3) = -d^\star \bar{a}_j, \tag{2.3}$$

where $\bar{a}_j = \bar{A}_j - \tilde{A}$. One can then see that

$$\begin{aligned}(g_1(t), a_1(t)) &:= ((\phi_1^i)_{t*}\bar{g}_1(t), (\phi_1^3)_{t*}\bar{a}_1(t))\\ (g_2(t), a_2(t)) &:= ((\phi_2^i)_{t*}\bar{g}_2(t), (\phi_2^3)_{t*}\bar{a}_2(t))\end{aligned}$$

are both solutions to equation (2.1). Notice that $(g_1(0), a_1(0)) = (g_2(0), a_2(0))$. Since equation (2.1) is parabolic, it has unique solutions, and so $(g_1(t), a_1(t)) = (g_2(t), a_2(t))$ for as long as the solutions exist. But then $(\phi_1^i)_t$ and $(\phi_2^i)_t$ are solutions to ODE generated by the same vector fields $W^i = -g^{pq}(\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k)$,

$i = 1, 2$ and $W^3 = -d^{\star}(A - \tilde{A})$. Hence, as long as they are defined, $(\phi_1^i)_t = (\phi_2^i)_t$. Thus $(\bar{g}_1, \bar{a}_1) = (\bar{g}_2, \bar{a}_2)$, and so solutions to the Ricci Yang-Mills Flow are unique.

Chapter 3

Long-time Behavior

3.1 Derivative Estimates and Long-time Existence

We would like to study aspects of the long-time behavior of the Ricci Yang-Mills flow. As in the case of Ricci flow, we establish derivative bounds for the Ricci Yang-Mills flow that hold for a short time, assuming certain curvature conditions. We then prove a long-time existence result analogous to that of Ricci flow. Finally, we prove a compactness theorem for the Ricci Yang-Mills flow and lay the groundwork for singularity analysis.

First, we would like to consider the following Bernstein-Bando-Shi derivative estimates.

Proposition 3.1.1. (A priori estimates) *Let $(M^2, g(t), a(t))$ be a solution to the Ricci Yang-Mills flow. For every $\alpha > 0$ and for every $m \in \mathbb{N}$, there exists a constant C_m depending on m, $\max\{\alpha, 1\}$, and K such that if*

$$|R(x,t)|_{g(x,t)}, |F(x,t)|^2_{g(x,t)} \leq K \textit{ for all } x \in M^2 \textit{ and } t \in [0, \frac{\alpha}{K}],$$

then

$$|\nabla^m F(x,t)|^2 + |\nabla^{m-1} R(x,t)|^2 \leq \frac{C_m}{t^m}, \textit{ for all } x \in M^2 \textit{ and } t \in (0, \frac{\alpha}{K}]. \quad (3.1)$$

In order to prove Proposition 3.1.1, we will need the following evolution equations. We use notation introduced in [5]; namely, that if A and B are any tensors, then $A * B$ denotes any quantity obtained from the tensor product of A and B and/or from contraction with the metric. By A^{*^n} we mean $A * \cdots * A$ n-times. Much of the analysis in this section is strongly motivated by that done for the Ricci flow in Chapter 7 of [5].

Lemma 3.1.2. *The following evolution equations hold:*

$$\partial_t R^2 = \Delta R^2 - 2|\nabla R|^2 + \nabla\nabla F * F * R + \nabla F^{*^2} * R + R^3 + R^{*^2} * F^{*^2},$$

$$\partial_t |\nabla F|^2 = \Delta |\nabla F|^2 - 2|\nabla\nabla F|^2 + R * \nabla F^{*^2} + \nabla R * F * \nabla F + F^{*^2} * \nabla F^{*^2}.$$

Proof. Using equation (1.9), we recall the evolution of R to be

$$\partial_t R = \Delta R + \Delta |F|^2 + R^2 - \frac{R}{2}|F|^2.$$

Then, notice that the term $\Delta|F|^2$ can be rewritten as $\nabla\nabla F * F + \nabla F * \nabla F$. So the first equation follows.

To compute the evolution of $|\nabla F|^2$, we notice first of all that

$$\partial_t \nabla F = \nabla \partial_t F + \partial_t \Gamma * F.$$

In our case, $\partial_t \Gamma = \nabla R + F * \nabla F$, as in equation 1.8, so we can compute

$$\begin{aligned}\partial_t |\nabla F|^2 &= 2\langle \partial_t \nabla F, \nabla F\rangle + \partial_t g * F^{*^2} \\ &= 2\langle \Delta \nabla F + R * \nabla F + \nabla R * F + F^{*^2} * \nabla F, \nabla F\rangle + (R + F^{*^2}) * \nabla F^{*^2} \\ &= \Delta |\nabla F|^2 - 2|\nabla\nabla F|^2 + R * \nabla F^{*^2} + \nabla R * F * \nabla F + F^{*^2} * \nabla F^{*^2}\end{aligned}$$

as required. □

Proof. *(of Proposition)* We will compute in detail our first BBS-estimate, and then sketch the proof for the higher derivatives. We compute the evolution of $|\nabla F|^2 + R^2$. In the computation, we would like to use the $-2|\nabla\nabla F|^2$ term to control the second derivative of F term that appears in the equation for R^2. Conversely, we use the $-2|\nabla R|^2$ term to control the corresponding term that appears in the equation for $|\nabla F|^2$. We accomplish this by repeated application of the Cauchy-Schwartz inequality. This will be our strategy throughout these estimates.

$$\begin{aligned}
\partial_t(|\nabla F|^2 + R^2) &\leq \Delta|\nabla F|^2 - 2|\nabla\nabla F|^2 + c|R||\nabla F|^2 + |\nabla R|^2 + c|F|^2|\nabla F|^2 \\
&+ \Delta R^2 - 2|\nabla R|^2 + |\nabla\nabla F|^2 + c|F|^2R^2 + c|R||\nabla F|^2 \\
&+ |R|^3 \\
&\leq \Delta(|\nabla F|^2 + R^2) - |\nabla\nabla F|^2 - |\nabla R|^2 + c|\nabla F|^2(|R| + |F|^2) \\
&+ c|F|^2R^2 + |R|^3,
\end{aligned}$$

where c depends only on the dimension and may change from line to line.

We do not assume an initial bound on $|\nabla F|^2$, so we define a quantity $G = t(|\nabla F|^2 + R^2) + \beta|F|^2$ and use the "good" $-\beta|\nabla F|^2$ term to control the "bad" term in the evolution of $|\nabla F|^2 + R^2$. Then we have that G satisfies

$$\partial_t G \leq \Delta G + |\nabla F|^2(ct|R| + ct|F|^2 + 1 - 2\beta) + ct|F|^2R^2 + t|R|^3 + \beta|R||F|^2 + \beta|F|^4.$$

By assumption, $|R|, |F|^2 \leq K$ on $[0, \frac{\alpha}{K})$, so on this interval, we have

$$\partial_t G \leq \Delta G + |\nabla F|^2(c\alpha + 1 - 2\beta) + cK^2\alpha + \beta K^3 + \beta K^2.$$

Then, by letting $\beta \geq \frac{c\alpha+1}{2}$, on the interval $[0, \frac{\alpha}{K}]$, we have

$$\partial_t G \leq \Delta G + c\beta K^2 + c\beta K^3.$$

So the maximum principle implies

$$\sup G(x,t) \leq \beta K + t(c\beta K^2 + c\beta K^3) \leq \beta K(1 + \alpha + \alpha K) \leq C_1.$$

Thus

$$|\nabla F|^2 + R^2 \leq \frac{C_1}{t^1},$$

on $[0, \frac{\alpha}{K})$ as required.

Now we would like to generalize this procedure to higher derivatives. First we would like to compute the evolution equations of $\nabla^k F$ and $\nabla^k R$. In both computations, we use the commutator formula

$$[\nabla^k, \Delta]A = \sum_{j=0}^{k} \nabla^j R * \nabla^{k-j} A,$$

where A is any tensor. As before c denotes some arbitrary constant that depends only on k and K and which may change from line to line.

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k F &= \nabla^k(\frac{\partial F}{\partial t}) + \sum_{j=0}^{k-1} \nabla^j(\nabla R + \nabla F * F) * \nabla^{k-1-j} F \\
&= \nabla^k \Delta F + \sum_{j=0}^{k} \nabla^j R * \nabla^{k-j} F + \sum_{j=0}^{k-1}\sum_{n=0}^{j} \nabla^{n+1} F * \nabla^{j-n} F * \nabla^{k-1-j} F \\
&= \Delta \nabla^k F + \sum_{j=0}^{k} \nabla^j R * \nabla^{k-j} F + \sum_{j=0}^{k-1}\sum_{n=0}^{j} \nabla^{n+1} F * \nabla^{j-n} F * \nabla^{k-1-j} F.
\end{aligned}$$

Similarly

$$\begin{aligned}
\frac{\partial}{\partial t}\nabla^k R &= \nabla^k(\frac{\partial R}{\partial t}) + \sum_{j=0}^{k-1}\nabla^j(\nabla R + \nabla F * F) * \nabla^{k-1-j}R \\
&= \nabla^k(\Delta R + \nabla\nabla F * F + \nabla F^{*^2} + R^2 + R * F^{*^2}) \\
&+ \sum_{j=0}^{k-1}\nabla^j(\nabla R + \nabla F * F) * \nabla^{k-1-j}R \\
&= \Delta\nabla^k R + \sum_{j=0}^{k}\nabla^j R * \nabla^{k-j}R + \sum_{j=0}^{k}\nabla^{k+2-j}F * \nabla^j F \\
&+ \sum_{j=0}^{k}\nabla^{j+1}F * \nabla^{k+1-j}F + \sum_{j=0}^{k}\sum_{n=0}^{k-j}\nabla^j R * \nabla^n F * \nabla^{k-j-n}F.
\end{aligned}$$

We can then use these formulas to compute the evolution equations for $|\nabla^k F|^2$ and $|\nabla^k R|^2$.

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k F|^2 &= \langle \partial_t \nabla^k F, \nabla^k F\rangle + (R + F^{*^2}) * (\nabla^k F)^{*^2} \\
&= \Delta|\nabla^k F|^2 - 2|\nabla^{k+1}F|^2 + \sum_{j=0}^{k}\nabla^j R * \nabla^{k-j}F * \nabla^k F \\
&+ \sum_{j=0}^{k-1}\sum_{n=0}^{j}\nabla^{n+1}F * \nabla^{j-n}F * \nabla^{k-1-j}F * \nabla^k F.
\end{aligned}$$

Then $|\nabla^k F|^2$ satisfies the differential inequality

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k F|^2 &\leq \Delta|\nabla^k F|^2 - 2|\nabla^{k+1}F|^2 + \sum_{j=0}^{k}c|\nabla^j R||\nabla^{k-j}F||\nabla^k F| \\
&+ \sum_{j=0}^{k-1}\sum_{n=0}^{j}c|\nabla^{n+1}F||\nabla^{j-n}F||\nabla^{k-1-j}F||\nabla^k F|.
\end{aligned}$$

Similarly, we have

$$\frac{\partial}{\partial t}|\nabla^k R|^2 = \Delta|\nabla^k R|^2 - 2|\nabla^{k+1}R|^2 + \sum_{j=0}^{k} \nabla^j R * \nabla^{k-j} R * \nabla^k R$$
$$+ \sum_{j=0}^{k} \nabla^{k+2-j} F * \nabla^j F * \nabla^k R$$
$$+ \sum_{j=0}^{k} \nabla^{j+1} F * \nabla^{k+1-j} F * \nabla^k R$$
$$+ \sum_{j=0}^{k}\sum_{n=0}^{k-j} \nabla^j R * \nabla^n F * \nabla^{k-j-n} F * \nabla^k R,$$

so that

$$\frac{\partial}{\partial t}|\nabla^k R|^2 \leq \Delta|\nabla^k R|^2 - 2|\nabla^{k+1}R|^2 + \sum_{j=0}^{k} c|\nabla^j R||\nabla^{k-j} R||\nabla^k R|$$
$$+ \sum_{j=0}^{k} c|\nabla^{k+2-j} F||\nabla^j F||\nabla^k R|$$
$$+ \sum_{j=0}^{k} c|\nabla^{j+1} F||\nabla^{k+1-j} F||\nabla^k R|$$
$$+ \sum_{j=0}^{k}\sum_{n=0}^{k-j} c|\nabla^j R||\nabla^n F||\nabla^{k-j-n} F||\nabla^k R|.$$

Now we would like to use induction to prove that equation (3.1) holds for $|\nabla^N F|^2 + |\nabla^{N-1} R|^2$. Suppose that on the time interval $(0, \frac{\alpha}{K}]$, we have

$$k \in 1 \leq \ldots \leq N-1,\ |\nabla^k F|^2 \leq \frac{C_k}{t^k}$$

and for

$$k \in 1 \leq \ldots \leq N-2,\ |\nabla^k R|^2 \leq \frac{C_{k+1}}{t^{k+1}}.$$

Notice that for such k, the following inequalities hold:

$$\frac{\partial}{\partial t}|\nabla^k F|^2 \leq \Delta|\nabla^k F|^2 - 2|\nabla^{k+1}F|^2 + \frac{c}{t^{k+\frac{1}{2}}} + \frac{c}{t^k},$$

and

$$\frac{\partial}{\partial t}|\nabla^k R|^2 \leq \Delta|\nabla^k R|^2 - 2|\nabla^{k+1}R|^2 + \frac{c}{t^{k+1}} + \frac{c}{t^{k+\frac{1}{2}}}.$$

As we did previously, we will use Cauchy-Schwartz and the $-2|\nabla^{N+1}F|^2$ and $-2|\nabla^N R|^2$ terms to bound the corresponding term appearing in the evolution of $|\nabla^{N-1}R|^2$ and $|\nabla^N F|^2$, respectively. Then we have

$$\begin{aligned}
\frac{\partial}{\partial t}(|\nabla^N F|^2 + |\nabla^{N-1}R|^2) \leq\ & \Delta(|\nabla^N F|^2 + |\nabla^{N-1}R|^2) - 2|\nabla^{N+1}F|^2 - 2|\nabla^N R|^2 \\
& + \frac{c}{t^{\frac{N+1}{2}}}|\nabla^N F| + c|R||\nabla^N F|^2 + c|\nabla^{N-1}R||\nabla F||\nabla^N F| \\
& + c|\nabla^N R||F||\nabla^N F| + \frac{c}{t^{\frac{N}{2}}}|\nabla^N F| + \frac{c}{t^{\frac{N+1}{2}}}|\nabla^{N-1}R| \\
& + |R||\nabla^{N-1}R|^2 + c|\nabla^{N+1}F||F||\nabla^{N-1}R| \\
& + c|\nabla^N F||\nabla F||\nabla^{N-1}R| + \frac{c}{t^{\frac{N}{2}}}|\nabla^{N-1}R| + \frac{c}{t^{\frac{N-1}{2}}}|\nabla^{N-1}R| \\
\leq\ & \Delta(|\nabla^N F|^2 + |\nabla^{N-1}F|^2) - 2|\nabla^{N+1}F|^2 - 2|\nabla^N R|^2 \\
& + \frac{c}{t^{\frac{N+1}{2}}}|\nabla^N F|^2 + c|R||\nabla^N F|^2 + \frac{c}{t}|\nabla^N F|^2 \\
& + c|\nabla^{N-1}R|^2 + |\nabla^N R|^2 + c|\nabla^N F|^2 \\
& + \frac{c}{t^{\frac{N}{2}}}|\nabla^N F| + \frac{c}{t^{\frac{N+1}{2}}}|\nabla^{N-1}R| + |R||\nabla^{N-1}R|^2 + |\nabla^{N+1}F|^2 \\
& + c|\nabla^{N-1}R|^2 + \frac{c}{t}|\nabla^N F|^2 + c|\nabla^{N-1}R|^2 + \frac{c}{t^{\frac{N}{2}}}|\nabla^{N-1}R| \\
& + \frac{c}{t^{\frac{N-1}{2}}}|\nabla^{N-1}R|
\end{aligned}$$

Combining like terms, we see that

$$\begin{aligned}\frac{\partial}{\partial t}(|\nabla^N F|^2 + |\nabla^{N-1}R|^2) \leq \Delta(|\nabla^N F|^2 + |\nabla^{N-1}R|^2) - |\nabla^{N+1}F|^2 - |\nabla^N R|^2 \\ + |\nabla^{N-1}R|(\frac{c}{t^{\frac{N+1}{2}}} + \frac{c}{t^{\frac{N}{2}}} + \frac{c}{t^{\frac{N-1}{2}}}) \\ + c|\nabla^{N-1}R|^2 + \frac{c}{t}|\nabla^N F|^2 + c|\nabla^N F|^2 \\ + |\nabla^N F|(\frac{c}{t^{\frac{N+1}{2}}} + \frac{c}{t^{\frac{N}{2}}}).\end{aligned}$$

Now we would like to define G to be

$$G = t^N(|\nabla^N F|^2 + |\nabla^{N-1}R|^2) + \beta_N \sum_{k=1}^{N-1} t^{N-k}(|\nabla^{N-k}F|^2 + |\nabla^{N-k-1}R|^2) + \beta_1|F|^2.$$

Then using the above computations, we see that G satisfies

$$\begin{aligned}\frac{\partial G}{\partial t} &\leq \Delta G - t^N|\nabla^N R|^2 - t^N|\nabla^{N+1}F|^2 + c|\nabla^{N-1}R|^2(t^{N-1} + t^N + t^{N+1}) \\ &+ c|\nabla^N F|^2(t^{N-1} + t^N) \\ &+ \beta_N \sum_{k=1}^{N}((N-k)t^{N-k-1}(|\nabla^{N-k}F|^2 + |\nabla^{N-k-1}R|^2) \\ &+ t^{N-k}(\partial_t(|\nabla^{N-k}F|^2 + |\nabla^{N-k-1}R|^2))) + \beta_1\partial_t|F|^2 + c.\end{aligned}$$

We use the good $t^{k-1}(|\nabla^k F|^2 + |\nabla^{k-1}R|^2)$ terms to control those bad terms above for all $k \leq N$. In a manner exactly analogous to the single derivative case, we obtain, on the interval $(0, \frac{\alpha}{K}]$, an estimate of the form

$$\partial_t G \leq \Delta G + (C - \beta_N)t^{N-1}|\nabla^N F|^2 + (C - \beta_N)t^{N-1}|\nabla^{N-1}R|^2 + c.$$

So we choose β_N large enough to make those terms negative, and we obtain

$$\partial_t G \leq \Delta G + c.$$

Notice that our assumption on $|F|^2$ implies that G is bounded at $t = 0$. Then using the maximum principle, we have

$$\sup_{x \in M} G \leq c + ct \leq C_N$$

on $[0, \frac{\alpha}{K})$, where C_N depends on N, K and the dimension. Thus,

$$|\nabla^N F|^2 + |\nabla^{N-1} R|^2 \leq \frac{C_N}{t^N}$$

on the same time interval, and our proposition is proved. □

We can extend these estimates to obtain bounds on the curvatures and all of their derivatives on a finite time interval.

Corollary 3.1.3. *Let $(M^2, g(t), a(t))$ be a solution of the Ricci Yang-Mills flow. If there exists $\beta > 0$ and $K > 0$ such that $|R|(x,t) \leq K$ and $|F|^2(x,t) \leq K$ for all $x \in M^2$ and $t \in [0, T)$, where $T > \frac{\beta}{K}$, then for all $m \in \mathbb{N}$, there exists a constant C_m depending only on m and $\min(\beta, 1)$ such that*

$$|\nabla^m F|^2 + |\nabla^{m-1} R|^2 \leq C_m K^m,$$

for all $x \in M^2$ and $t \in [\frac{\min(\beta,1)}{K}, T]$.

Proof. The proof follows that in [5]. Let $\beta_1 := \min(\beta, 1)$. Let $t_0 \in [\frac{\beta_1}{K}, T]$ be arbitrary, and define $T_0 := t_0 - \frac{\beta_1}{K}$. Additionally, let $\bar{t} = t - T_0$. We define

$(\bar{g}(\bar{t}), \bar{a}(\bar{t}))$ to be the solution of the initial value problem

$$\begin{aligned} \frac{\partial \bar{g}}{\partial \bar{t}} &= -\bar{R}\bar{g} + \frac{1}{2}|\bar{F}|^2\bar{g} \\ \frac{\partial \bar{a}}{\partial \bar{t}} &= -d^\star \bar{F} \\ \bar{g}(0) &= g(T_0) \\ \bar{a}(0) &= a(T_0). \end{aligned}$$

Since solutions to the Ricci Yang-Mills flow are unique, $\bar{g}(\bar{t}) = g(\bar{t}+T_0) = g(t)$ and $\bar{a}(\bar{t}) = a(\bar{t} + T_0) = a(t)$ for $t \in [0, \frac{\beta_1}{K}]$. We assume that $|\bar{R}| \leq K$ and $|\bar{F}|^2 \leq K$ for all $x \in M^2$ and $\bar{t} \in [0, \frac{\beta_1}{K}]$, so that we can apply our BBS estimates with $\alpha = \beta_1$. Then we are guaranteed constants $\bar{C}_m$ such that

$$|\bar{\nabla}^m \bar{F}|^2_{\bar{g}} + |\bar{\nabla}^{m-1}\bar{R}|^2_{\bar{g}} \leq \frac{\bar{C}_m}{t^m},$$

for all $x \in M^2$ and $\bar{t} \in [0, \frac{\beta_1}{K}]$. Notice that for $\bar{t} \in [\frac{\beta_1}{2K}, \frac{\beta_1}{K}]$, we have $\bar{t}^m \geq \frac{\beta_1^m}{2^m K^m}$. Taking $\bar{t} = \frac{\beta_1}{K}$, we see that

$$|\nabla^m F(x, t_0)|^2 + |\nabla^{m-1} R(x, t_0)|^2 \leq \frac{2^m K^m \bar{C}_m}{\beta_1^m},$$

for all $x \in M^2$. The result follows, since t_0 was arbitrary. □

We would also like to derive a quadrupling time estimate analogous to the doubling time estimate of Ricci flow. The difference is the exponent in the applicable evolution equation. Recall the form of the Ricci tensor on the total space of the bundle. It contains the terms R, $|F|^2$, and $d^\star F$. So we will define Φ to be a linear combination of R^2, $|F|^4$, and $|d^\star F|^2$, and we will estimate Φ. Notice that this is equivalent to estimating $|Rc|^2$.

Lemma 3.1.4. *Let* $\Phi := R^2 + 2|d^\star F|^2 + 2|F|^4$*. Then*

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + C\Phi^{\frac{3}{2}}.$$

Proof. Recall the evolution equations:

$$\begin{aligned}
\partial_t R &= \Delta R - \frac{1}{2}\Delta|F|^2 + R^2 - \frac{R}{2}|F|^2 \\
\partial_t |d^* F|^2 &= \Delta|d^*F|^2 - 2|\nabla d^* F|^2 + 2R|d^*F|^2 - \frac{3}{2}|F|^2|d^*F|^2 \\
&\quad + 2\langle \nabla^j R F_{jk}, d^* F_k\rangle - \langle \nabla^j |F|^2 F_{jk}, d^* F_k\rangle \\
\partial_t |F|^4 &= \Delta|F|^4 - 2|\nabla|F|^2|^2 - 2|\nabla F|^2|F|^2 + 4R|F|^4 - 2|F|^6.
\end{aligned}$$

We would like to estimate the term $\langle \nabla^j |F|^2 F_{jk}, d^* F_k\rangle$ as follows:

$$\begin{aligned}
\langle \nabla^j |F|^2 F_{jk}, d^* F_k\rangle &\leq \frac{|\nabla|F|^2|^2}{2} + \frac{|F|^2|d^*F|^2}{2} \\
&\leq 2|\nabla F|^2|F|^2 + \frac{|F|^2|d^*F|^2}{2}.
\end{aligned}$$

Then (estimating the other bad terms as in the previous theorem), we have

$$\partial_t \Phi \leq \Delta\Phi - 2R|\nabla F|^2 + 2R^3 + 4R|d^*F|^2 + 4R|F|^4 - 4|F|^6.$$

Using the standard inequality that $|Z|^2 \geq \frac{1}{n}|tr_g Z|^2$, we find

$$\begin{aligned}
\partial_t \Phi &\leq \Delta\Phi + 2R^3 + 2R|d^*F|^2 + 4R|F|^4 \\
&\leq \Delta\Phi + |R|^3 + |R||\Phi| + 2|R||F|^4 \\
&\leq \Delta\Phi + |\Phi|^{\frac{3}{2}},
\end{aligned}$$

as required. □

Corollary 3.1.5. (Quadrupling-time Estimate) *There exists a* $c > 0$ *such that if* $(M^2, g(t), a(t))$ *is a solution of the Ricci Yang-Mills flow on* $[0, \tau)$ *and*

$$M(t) := \sup_{x \in M^2} \Phi(x,t),$$

then

$$M(t) \leq 4M(0) \text{ for all times } 0 \leq t \leq \min\{\tau, \frac{c}{\sqrt{M(0)}}\}.$$

Proof. Notice that $M(t)$ is a Lipschitz function of time. In the sense of forward difference quotients, Lemma 3.1.4 implies that M satisfies

$$\frac{dM}{dt} \leq CM^{\frac{3}{2}}.$$

Then

$$M(t) \leq \frac{1}{(\frac{1}{\sqrt{M(0)}} - 2Ct)^2},$$

if $t \leq \frac{1}{2C\sqrt{M(0)}}$. Let $c = \frac{1}{2}C$. Then for $t \in [0, \min\{\tau, \frac{c}{\sqrt{M(0)}}\})$,

$$M(t) \leq 4M(0).$$

□

We can then use the previous facts to give an obstruction to the long time existence for the Ricci Yang-Mills flow. Just as in the case of Ricci flow, this obstruction comes in the form of a curvature blow-up.

Theorem 3.1.6. *Let* M^2 *be a compact manifold. If* g_0 *is a smooth metric and* a_0 *is a smooth connection 1-form, then the unique solution* $(g(t), a(t))$ *of* *the Ricci Yang-Mills flow such that* $(g(0), a(0)) = (g_0, a_0)$ *exists on a maximal time interval* $0 \le t < T \le \infty$. *Moreover, if* $T < \infty$, *then*

$$\lim_{t \to T} (\sup_{x \in M^2} |\Phi|(x,t)) = \infty.$$

Proof. Let $M(t)$ as above. We would first like to show that if T is finite, then

$$\limsup_{t \to T} M(t) = \infty.$$

As in the proof for Ricci flow, we will prove the contrapositive of this statement. In particular, suppose that a solution to RYM exists on a maximal finite time interval and that there exists a $K \ge 0$ such that $\sup_{0 \le t < T} M(t) \le K$. Fix a local coordinate patch $\mathcal{U}$ around an arbitrary point $x \in M^2$ and let $\tau \in (0, T)$ be arbitrary. By Lemma 6.49 in [5], a continuous limit metric $g(T)$ exists and is given by

$$g_{ij}(x,T) = g_{ij}(x,\tau) + \int_{\tau}^{T} (-Rg_{ij}(x,t) + \frac{1}{2}|F|^2 g_{ij}(x,t))dt.$$

Similarly, a continuous limit connection 1-form $a(T)$ exists and is given by

$$a_i(x,T) = a_i(x,\tau) - \int_{\tau}^{T} d^{\star} F_i(x,t)dt.$$

Let $m \in \mathbb{N}$, and choose α to be a multi-index such that $|\alpha| = m$. Since Γ is bounded by Corollary 3.1.3, it is straightforward to show that all of $\frac{\partial^m}{\partial x^\alpha} g_{ij}$, $\frac{\partial^m}{\partial x^\alpha} R$, $\frac{\partial^m}{\partial x^\alpha} F_{ij}$, and $\frac{\partial^m}{\partial x^\alpha} a_i$ are uniformly bounded on $\mathcal{U} \times [0, T)$. From the integral formulations of g and a above, we see that

$$\frac{\partial^m}{\partial x^\alpha} g_{ij}(x,T) = \frac{\partial^m}{\partial x^\alpha} g_{ij}(x,\tau) + \int_{\tau}^{T} (-\frac{\partial^m}{\partial x^\alpha}(Rg_{ij})(x,t) + \frac{1}{2}\frac{\partial^m}{\partial x^\alpha}(|F|^2 g_{ij})(x,t)]dt,$$

and

$$\frac{\partial^m}{\partial x^\alpha} a_i(x,T) = \frac{\partial^m}{\partial x^\alpha} a_i(x,\tau) - \int_\tau^T \frac{\partial^m}{\partial x^\alpha} d^\star F_i(x,t) dt.$$

This implies that $(g(T), a(T))$ is smooth and that $(g(\tau), a(\tau)) \to (g(T), a(T))$ in any C^m norm as $t \to T$. Thus, the short time existence result implies that there exists a solution $(\bar{g}(t), \bar{a}(t))$ of RYM such that $\bar{g}(0) = g(T)$ and $\bar{a}(0) = a(T)$ for $0 \le t < \epsilon$. Since $(g(\tau), a(\tau)) \to (g(T), a(T))$ smoothly, we can extend the solution past T with the same initial data (g_0, a_0). This contradicts the maximality of T.

Now we would like to replace the lim sup with an actual limit. Again, suppose the theorem is false. Then there exists $K_0 \le \infty$ and a sequence of times $t_i \nearrow T$ such that $M(t_i) \le K_0$. Corollary 3.1.5 implies that there exists a $c > 0$ such that $M(t) \le 4M(t_i) \le 4K_0$ for all t such that $t_i \le t < \min(T, t_i + \frac{C}{K_0})$. As $i \to \infty$, $t_i \nearrow T$, so there is i_0 such that $t_{i_0} + \frac{C}{K_0} \ge T$. But then

$$\sup_{t_{i_0} \le t < T} M(t) \le 4K_0,$$

which is a contradiction. Thus our full theorem is proved. □

We would like to prove a final lemma that shows that bounds on R on a finite time interval imply bounds on $|F|^2$.

Lemma 3.1.7. *Let $T < \infty$ and suppose that $|R| \le C$ on $[0,T]$. Then $\sup_{M\times[0,T]} |F|^2 \le \tilde{C}$, where $\tilde{C}$ depends on C and $|F|^2(0)$.*

Proof. We recall the evolution of $|F|^2$ given in equation (1.11):

$$\partial_t |F|^2 = \Delta |F|^2 - 2|\nabla F|^2 + 2R|F|^2 - |F|^4.$$

Define $M(t) := \sup_{x \in M} |F|^2(t)$. Then, if $|R| \leq C$, M satisfies (in the sense of forward difference quotients)

$$\frac{dM}{dt} \leq 2CM,$$

so that $M(t) \leq M(0)e^{2Ct} \leq M(0)e^{2cT}$, as required. □

3.2 Compactness

One would like to be able to study the singularity formation of the Ricci Yang-Mills flow and to compare and contrast it to that of Ricci flow. The monotonicity formulas in Chapter 4 provide evidence that such analysis may be possible. Here we lay the groundwork by proving the analogue of Hamilton's compactness theorem [11] for the Ricci Yang-Mills flow. Notice that quantities that are measured with respect to (g_k, a_k) will be denoted with either a subscript or superscript k. Quantities without such demarcation are assumed to be measured with respect to a fixed background metric and connection.

We begin by proving a lemma analogous to Lemma 2.4 in [11].

Lemma 3.2.1. *Let (M^2, g) be a Riemannian manifold and $K \subset M^2$ a compact subset. Suppose $(g_k(t), a_k(t))$ is a one-parameter family of solutions to the Ricci Yang-Mills flow defined on neighborhoods of $[\alpha, \omega] \times K$ such that $\alpha < 0 < \omega$. At $t = 0$ on K, let*

i. $cg(X,X) \leq g_k(X,X) \leq Cg(X,X)$,

ii. $|\nabla^p g_k| \leq C_p$, *for all* $p \geq 1$,

iii. $|\nabla^p F_k| \leq C'_p$, *for all* $p \geq 0$.

Assume also

iv. $\sup_{[\alpha,\omega]\times K} |{}^k\nabla^p R_k|_k \leq C_p$ *for all* $p \geq 0$,

v. $\sup_{[\alpha,\omega]\times K} |{}^k\nabla^p F_k|_k \leq C'_p$ *for all* $p \geq 0$.

Then the following holds:

a. $\tilde{c}g(X,X) \leq g_k(X,X) \leq \tilde{C}g(X,X)$, *on* $[\alpha,\omega]\times K$

b. $\sup_{[\alpha,\omega]\times K} |\nabla^p g_k| \leq C_p$, *for all* $p \geq 1$,

c. $\sup_{[\alpha,\omega]\times K} |\nabla^p F_k| \leq C'_p$, *for all* $p \geq 0$.

Proof. Let X be a vector field on M^2. For all k, we have

$$\begin{aligned}\partial_t g_k(X,X) &= -R_k|X|_k^2 + \frac{1}{2}|F_k|_k^2|X|_k^2 \\ &\leq A_0|X|_k^2,\end{aligned}$$

by assumptions (iv) and (v). Then

$$\partial_t \ln g_k(X,X) = \frac{1}{g_k(X,X)}\partial_t g_k(X,X) \leq A_0,$$

so $|\partial_t \ln g_k(X,X)| \leq A_0$. Throughout the proof of this lemma, we will let $0 < t < \omega$ be arbitrary. Then we integrate to obtain

$$\begin{aligned}\ln g_k(X,X)(t) &\leq \ln g_k(X,X)(0) + \int_0^t \partial_\tau \ln g_k(X,X) d\tau \\ &\leq \ln g_k(X,X)(0) + A_0\omega.\end{aligned}$$

Then we exponentiate to see that

$$g_k(t) \leq e^{A_0\omega} g_k(0) \leq e^{A_0\omega} C g =: \tilde{C} g,$$

where we use assumption (i). We can similarly show that $g_k(X,X) \geq \tilde{c} g$, and so (a) is proved. Next we recall the evolution of the Christoffel symbols Γ. Namely, we have

$$\partial_t(\Gamma_k - \Gamma) = {}^k\nabla\, R + F_k * {}^k\nabla\, F_k.$$

Then

$$|\partial_t(\Gamma_k - \Gamma)| \leq C|\, {}^k\nabla\, R|_k + C|F_k|_k|\, {}^k\nabla\, F_k| \leq C =: A_1,$$

by assumptions (iv) and (v). Since $\nabla g_k \simeq \Gamma_k - \Gamma \simeq {}^k\nabla - \nabla$, we can bound $|\nabla g_k|$ by

$$|\partial_t \nabla g_k| \leq c|\partial_t(\Gamma_k - \Gamma)| \leq cC'|\partial_t(\Gamma_k - \Gamma)|_k \leq C' A_1,$$

where C' comes from (a). Integrating again yields

$$\begin{aligned}|\nabla g_k|(t) &= |\nabla g_k(0) + \int_0^t \partial_\tau \nabla g_k d\tau| \\ &\leq |\nabla g_k(0)| + C' A_1 \omega \\ &\leq \tilde{C}_1 + C' A_1 \omega =: \tilde{C}_1,\end{aligned}$$

where the third inequality follows from assumption (ii). Now consider $|F_k|$. By (a) and (v), we have

$$|F_k|(t) \leq C'|F_k|_k(t) \leq \tilde{C}',$$

on $[\alpha, \omega]$. Using the fact that ∇ is independent of time, we have

$$\begin{aligned}\partial_t \nabla F_k = \nabla \partial_t F_k &= \nabla \ {}^k\Delta \ F_k \\ &= (\nabla - {}^k\nabla) \ {}^k\Delta \ F_k + {}^k\nabla \ {}^k\Delta \ F_k.\end{aligned}$$

Then by (b) and (v), we have

$$|\partial_t \nabla F_k| \leq C|\nabla g_k|| \ {}^k\Delta \ F_k|_k + C| \ {}^k\nabla \ {}^k\Delta \ F_k| \leq C.$$

As above,

$$|\partial_t \nabla F_k|(t) \leq |\nabla F_k|(0) + \int_0^t |\partial_\tau \nabla F_k| d\tau \leq C_1'.$$

In a similar fashion, we write

$$\begin{aligned}\partial_t \nabla^2 g_k = \nabla^2 \partial_t g_k &= \nabla^2(-R_k g_k + \frac{1}{2}|F_k|_k^2) \\ &= \nabla^2 R_k + \nabla^2 F_k * F_k + \nabla F_k * \nabla F_k \\ &+ \nabla R_k * \nabla g_k + R_k * \nabla^2 g_k + F_k^{*^2} * \nabla^2 g_k \\ &+ \nabla F_k * F_k * \nabla g_k.\end{aligned}$$

We can rewrite some of these terms to be

$$\nabla^2 R_k = (\nabla - {}^k\nabla) dR_k + {}^k\nabla \ dR_k = \nabla g_k * {}^k\nabla \ R_k + {}^k\nabla^2 \ R_k,$$

$$\nabla^2 F_k = \nabla g_k * \nabla g_k * F_k + \nabla g_k * {}^k\nabla \ F_k + {}^k\nabla \ \nabla g_k * F_k + \nabla g_k * {}^k\nabla \ F_k + {}^k\nabla^2 \ F_k,$$

$$ {}^k\nabla\,\nabla g_k = \nabla^2 g_k + ({}^k\nabla - \nabla)\nabla g_k = \nabla^2 g_k + \nabla g_k * \nabla g_k, $$

$$ \begin{aligned} \nabla F_k * \nabla F_k &= ((\nabla - {}^k\nabla)F_k + {}^k\nabla\, F_k) * ((\nabla - {}^k\nabla)F_k + {}^k\nabla\, F_k) \\ &= (\nabla g_k * F_k + {}^k\nabla\, F_k) * (\nabla g_k * F_k + {}^k\nabla\, F_k) \\ &= \nabla g_k * \nabla g_k * F_k * F_k + \nabla g_k * {}^k\nabla\, F_k * F_k + {}^k\nabla\, F_k * {}^k\nabla\, F_k. \end{aligned} $$

Combining all of these terms, we can see that

$$ \begin{aligned} \partial_t \nabla^2 g_k &= \nabla g_k * {}^k\nabla\, R_k + {}^k\nabla^2\, R_k \\ &+ (\nabla g_k * \nabla g_k * F_k + \nabla g_k * {}^k\nabla\, F_k + \nabla^2 g_k * F_k + \nabla g_k * \nabla g_k * F_k \\ &+ {}^k\nabla^2\, F_k) * F_k + \nabla g_k * \nabla g_k * F_k * F_k + \nabla g_k * F_k * {}^k\nabla\, F_k \\ &+ {}^k\nabla\, F_k * {}^k\nabla\, F_k + {}^k\nabla\, R_k * \nabla g_k + R_k * \nabla^2 g_k \\ &+ F_k^{*^2} * \nabla^2 g_k + \nabla g_k^{*^2} * F_k^{*^2} \\ &= R_k * \nabla^2 g_k + F_k^{*^2} * \nabla^2 g_k + {}^k\nabla\, R_k * \nabla g_k \\ &+ {}^k\nabla^2\, R_k + \nabla g_k^{*^2} * F_k^{*^2} + \nabla g_k * {}^k\nabla\, F_k * F_k \\ &+ {}^k\nabla^2\, F_k * F_k + {}^k\nabla\, F_k * {}^k\nabla\, F_k. \end{aligned} $$

Thus

$$ \begin{aligned} |\partial_t \nabla^2 g_k| &\le c|R_k||\nabla^2 g_k| + c|F_k|^2|\nabla^2 g_k| + c|\nabla g_k||\,{}^k\nabla\, R_k| + c|\,{}^k\nabla^2\, R_k| \\ &+ c|\nabla g_k|^2|F_k|^2 + |\nabla g_k||\,{}^k\nabla\, F_k||F_k| + |\nabla^2 g_k||F_k|^2 + |\,{}^k\nabla^2\, F_k||F_k| \\ &+ |\,{}^k\nabla\, F_k|^2 \end{aligned} $$

Then we have that $|\partial_t \nabla^2 g_k| \le C|\nabla^2 g_k| + C$, and since $|\nabla^2 g_k|(0)$ is bounded, we can integrate to obtain

$$ |\nabla^2 g_k|(t) \le C_2 $$

on $[\alpha, \omega]$. Similar to the computations above, we have

$$\partial_t \nabla^2 F_k = \nabla^2 \partial_t F_k = \nabla^2 \ {}^k\Delta \ F_k.$$

We can rewrite $\nabla^2 \ {}^k\Delta$ to be

$$\begin{aligned}\nabla^2 \ {}^k\Delta \ F_k = (\nabla g_k)^{*^2} * {}^k\Delta \ F_k + \nabla g_k * {}^k\nabla \ {}^k\Delta \ F_k \\ + \nabla^2 g_k * {}^k\Delta \ F_k + {}^k\nabla^2 \ {}^k\Delta \ F_k.\end{aligned}$$

Then

$$\begin{aligned}|\partial_t \nabla^2 F_k| &\leq |\nabla g_k|^2 | \ {}^k\Delta \ F_k| + |\nabla g_k| | \ {}^k\nabla \ {}^k\Delta \ F_k| \\ &+ |\nabla^2 g_k| | \ {}^k\Delta \ F_k| + | \ {}^k\nabla^2 \ {}^k\Delta \ F_k| \\ &\leq C| \ {}^k\nabla^2 \ F_k| + C| \ {}^k\nabla^3 \ F_k| + c| \ {}^k\nabla^4 \ F_k| \\ &\leq C.\end{aligned}$$

Integrating as before implies that $|\nabla^2 F_k| \leq C_2'$. We would like to derive some recursion formulas for higher derivatives.

$$\begin{aligned}\partial_t \nabla^p g_k &= \nabla^p(-R_k g_k + \frac{1}{2}|F_k|_k^2 g_k) \\ &= R_k * \nabla^p g_k + F_k^{*^2} * \nabla^p g_k + {}^k\nabla^p \ R_k + {}^k\nabla^p \ |F_k|_k^2 \\ &+ \sum_{n=1}^{p-1} {}^k\nabla^n \ R_k * \nabla^{p-n} g_k \\ &+ \sum_{n=1}^{p-1} \sum_{m=0}^{n} {}^k\nabla^m \ F_k * {}^k\nabla^{n-m} \ F_k * \nabla^{p-n} g_k.\end{aligned}$$

Then $|\partial_t \nabla^p g_k|$ satisfies the following differential inequality.

$$
\begin{aligned}
|\partial_t \nabla^p g_k| \leq c|R_k||\nabla^p g_k| &+ c|F_k|^2|\nabla^p g_k| + c|\,{}^k\nabla^p\, R_k| \\
&+ c\sum_{n=0}^{p} |\,{}^k\nabla^n\, F_k||\,{}^k\nabla^{p-n}\, F_k| + \sum_{n=1}^{p-1} c|\,{}^k\nabla^n\, R_k||\nabla^{p-n} g_k| \\
&+ \sum_{n=1}^{p-1}\sum_{m=0}^{n} c|\,{}^k\nabla^m\, F_k||\,{}^k\nabla^{n-m}\, F_k||\nabla^{p-n} g_k|.
\end{aligned}
$$

By induction, we have that $|\nabla^n g_k|$ bounded for all $n < p$. Then we obtain

$$|\partial_t \nabla^p g_k| \leq c|\nabla^p g_k| + c,$$

so integration yields $|\nabla^p g_k| \leq C_p$ on $[\alpha, \omega]$. We can also derive a formula for higher derivatives of F.

$$
\begin{aligned}
\partial_t \nabla^p F_k &= \nabla^p\, {}^k\Delta\, F_k \\
&= {}^k\nabla^p\, {}^k\Delta\, F_k + \sum_{n=0}^{p-1} {}^k\nabla^n\, {}^k\Delta\, F_k * {}^k\nabla^{p-1-n}\, \nabla g_k \\
&+ \sum_{n=0}^{p-2} \nabla g_k * {}^k\nabla^n\, \nabla g_k * {}^k\nabla^{p-2-n}\, {}^k\Delta\, F_k \\
&+ \sum_{n=0}^{p-3} P(\nabla g_k, {}^k\nabla^n\, \nabla g_k)\, {}^k\nabla^{p-3-n}\, {}^k\Delta\, F_k. \\
&= {}^k\nabla^p\, {}^k\Delta\, F_k + \sum_{n=0}^{p-1} P(\nabla g_k, \ldots, \nabla^{p-n} g_k) * {}^k\nabla^n\, {}^k\Delta\, F_k,
\end{aligned}
$$

where P is some polynomial. Then

$$
\begin{aligned}
|\partial_t \nabla^p F_k| &\leq c|\,{}^k\nabla^{p+2}\, F_k| + \sum_{n=0}^{p-1} c|\,{}^k\nabla^{p+2}\, F_k| \\
&\leq C_p',
\end{aligned}
$$

and one more integration yields the desired result. $\square$

When we study singularity formation, it is important to have a notion of convergence that preserves the interesting part of the geometry in the limit. As such, we want to consider convergence in the pointed category.

Definition 3.2.1. Let P be a principal $U(1)$ bundle with base manifold M, where M is a complete Riemannian manifold with metric $g(t)$. Let P have connection 1-form $a(t)$, and let $x \in M$ be a choice of basepoint. An ordered triple $(M, (g, a)(t), x)$ is said to be a *pointed solution to the Ricci Yang-Mills flow* if $(M, (g, a)(t))$ is a solution to the Ricci Yang-Mills flow.

Then we can define smooth Cheeger-Gromov convergence of solutions to the Ricci Yang-Mills flow.

Definition 3.2.2. Let $(g_k, a_k)(t)$ be a solution to the Ricci Yang-Mills flow on $[T_A, T_O) \times M_k$, where M_k is complete. Let x_k be a base point, and let M_∞ be a complete Riemannian manifold, (g_∞, a_∞) a solution, and x_∞ a base point. Then we say the ordered triple $(M_k, (g_k, a_k)(t), x_k)$ *converges to* $(M_\infty, (g_\infty, a_\infty)(t), x_\infty)$ if there exists a sequence of open sets $U_k \subset M_\infty$ containing x_∞ and a sequence of diffeomorphisms $\phi_k : U_k \to V_k$, where $V_k \subset M_k$ is open and $\phi_k(x_\infty) = x_k$, such that any compact set in M_∞ eventually lies in all of the U_k and $\tilde{g}_k(t) := \phi_k^* g_k(t)$ and $\tilde{a}_k(t) := \phi_k^* a_k(t)$ converge uniformly to $(g_\infty, a_\infty)(t)$ on every compact subset of $(T_A, T_O) \times M_\infty$ along with all derivatives.

Our compactness theorem will make use of the following theorem of Hamilton

[11]. In what follows, $inj_g(x)$ denotes the injectivity radius of g measured at the point x.

Theorem 3.2.2. (Hamilton's Compactness for Metrics) *Let* $\{(M_k, g_k, x_k)\}_{k\in\mathbb{N}}$ *be a sequence of complete pointed Riemannian manifolds such that*

$$|\,{}^k\nabla^p\, Rm_k|_k \leq C_p,$$

for all k *and all* $p \geq 0$*, where* $C_p < \infty$ *are independent of* k*, and*

$$inj_{g_k}(x_k) \geq \iota_0 > 0.$$

Then there exists a subsequence (also denoted with k*) such that* $\{(M_k, g_k, x_k)\}_{k\in\mathbb{N}}$ *converges to a complete pointed Riemannian manifold* $(M_\infty, g_\infty, x_\infty)$ *as* $k \to \infty$*.*

Now we can write the compactness theorem for the Ricci Yang-Mills flow.

Theorem 3.2.3. (Compactness) *Let* T_A, T_O *be given such that* $-\infty \leq T_A < 0 < T_O \leq \infty$*. Fix* $t_0 \in (T_A, T_O)$*. Let* $(M_k, (g_k, a_k)(t), x_k)$ *be a sequence of complete solutions to the Ricci Yang-Mills flow for* $t \in [T_A, T_O)$ *such that*

$$\sup_{M_k} |R_k|_k(t) \leq C_0, \text{ for all } t \in (T_A, T_O),$$

$$\sup_{M_k} |F_k|_k(T_A) \leq C_0'.$$

Suppose also that $inj_{g_k(0)}(x_k) \geq c_o > 0$ *at time* $t = 0$*. Then there exists a subsequence converging in the sense of Definition 3.2.2*

$$(M_k, (g_k(t), a_k(t)), x_k) \to (M_\infty, (g_\infty(t), a_\infty(t)), x_\infty)$$

to a complete solution of Ricci Yang-Mills flow such that all derivatives of $R(g_\infty)$ and $F(a_\infty)$ are bounded above and the injectivity radius is bounded below.

Proof. Let $(g_k, a_k)(t)$ be a sequence of solutions on $[T_A, T_O) \times M_k$ such that $|R_k|_k \leq C$ and $|F_k|_k^2(T_A) \leq C$. Assume for simplicity that both T_A and T_O are finite. We also assume that the injectivity radius is bounded below by some positive constant at time $t = 0$. Lemma 3.1.7 states that uniform bounds on $|R_k|_k$ and initial bounds on $|F_k|_k^2$ imply uniform bounds on $|F_k|_k^2$ on $[T_A, T_O)$. By Corollary 3.1.3, we see that our uniform curvature bounds imply uniform bounds on the derivatives of curvature; namely, $|\,{}^k\nabla^m F_k|_k^2 + |\,{}^k\nabla^{m-1} R_k|_k^2 \leq C_m$ for all $m \geq 0$. Using Theorem 3.2.2, we can then obtain a subsequence of $(M_k, g_k(0), x_k)$ that converges to a limit (M_∞, h, x_∞). In particular,

$$\lim_{k\to\infty} |\,{}^h\nabla^m (\phi_k^* g_k(0)) - {}^h\nabla^m h|_h = 0,$$

for all $m \geq 0$.

We would like to use Lemma 3.2.1 to prove the convergence of (g_k, a_k) to a limit for all t. We will define $\tilde{g}_k(t) := \phi_k^* g_k(t)$ and $\tilde{a}_k(t) := \phi_k^* a_k(t)$. Notice that these are defined on (T_A, T_O). Let $[\alpha, \omega] \subset (T_A, T_O)$ such that $0 \in [\alpha, \omega]$, and let $K \subset M_\infty$ be compact. Given that $\tilde{g}_k(0)$ converges to h at $t = 0$, we know that $\tilde{g}_k(0)$ is equivalent to h on K. Additionally, we know that the covariant derivatives of $\tilde{g}_k(0)$ with respect to h are uniformly bounded on $\{0\} \times K$. By assumption and the equivalence of metrics, $|\tilde{F}_k(0)|_h \leq C|F_k(0)|_k \leq C$.

We would like to show that there exist $C_m \leq \infty$ such that $| {}^h\nabla^m \tilde{a}_k|_k \leq C_m$ for all $m \geq 0$ at $t = 0$. Notice that for $m \geq 1$, this is equivalent to showing $| {}^h\nabla^m \tilde{F}_k|_h \leq C'_m$, for some $C'_m \leq \infty$. Then we can use the equivalence of metrics and the fact that ${}^{\tilde{g}_k(0)}\nabla^m \to {}^h\nabla^m$ at $t = 0$ to obtain

$$\begin{aligned}
| {}^h\nabla^m \tilde{F}_k(0)|_h &\leq C| {}^h\nabla^m \tilde{F}_k(0)|_{\tilde{g}_k} \\
&\leq C| {}^{\tilde{g}_k(0)}\nabla^m \tilde{F}_k(0)|_{\tilde{g}_k(0)} \\
&\leq C| {}^k\nabla^m F_k(0)|_k \\
&\leq C'_m,
\end{aligned}$$

for k large enough. Additionally, since we have bounds on the curvature, we can find a corresponding sequence of connections such that $|a_k|_h \leq C'_0$. Then Lemma 3.2.1 yields $ch(t) \leq g_k(t) \leq Ch(t)$ on $[\alpha, \omega] \times K$ as well as uniform bounds on the derivatives of both g_k and F_k on $[\alpha, \omega] \times K$. We can then apply the Arzela-Ascoli theorem to get a convergent subsequence that converges uniformly on compact subsets of $(T_A, T_O) \times M_\infty$ to a limit. We define $(g_\infty, a_\infty) := (\lim_{k\to\infty} g_k, \lim_{k\to\infty} a_k)$. Notice that $g_\infty(0) = h(0)$ by definition. Additionally, (g_∞, a_∞) is a solution of RYM since the convergence is smooth. It also retains its curvature and injectivity radius bounds. $\square$

Chapter 4

Gradient Properties

4.1 Energy Monotonicity

Many well-known partial differential equations can be thought of as the gradient flow of some functional. For example, we saw in Chapter 1 that the Yang-Mills heat flow is the gradient flow of the Yang-Mills functional. On the other hand, it has long been known that Ricci flow is not the gradient flow of a functional on the space of metrics with respect to the standard L^2 norm. This is unfortunate as variational methods can be very powerful tools in analyzing PDE. In [21], Perelman shows that in fact the Ricci flow can be viewed as a gradient flow on a larger configuration space. We would like to follow the ideas of Perelman in order to write the Ricci Yang-Mills flow as a gradient flow. We claim that our coupled system is the gradient flow of some functional $\mathcal{F}(g, a, f)$ analogous to that of Perelman. We show that $\mathcal{F}$ obeys a certain monotonicity property along the flow. We also give a geometric consequence of this fact–namely that there are no non-trivial steady breathers.

We define

$$\mathcal{F} = \int_M (R - \frac{1}{4}|F|^2 + |\nabla f|^2) e^{-f} dV,$$

where f is some function on M to be determined.

We would like to compute the variation of $\mathcal{F}$. Let h, b, and s denote the variations of g, a, and f respectively.

The first lemma will be a useful fact that we will often use without mention.

Lemma 4.1.1.

$$\int \Delta f e^{-f} dV = \int |\nabla f|^2 e^{-f} dV.$$

Proof. This follows from the facts that

$$\int \Delta(e^{-f}) dV = 0$$

and

$$\Delta(e^{-f}) = (-\Delta f + |\nabla f|^2)e^{-f}.$$

□

Lemma 4.1.2.

$$\partial_t(e^{-f} dV) = (\frac{H}{2} - s)e^{-f} dV.$$

Proof. We use the variation formula for dV, namely that $\partial_t dV = \frac{H}{2} dV$, where $H = g^{ij} h_{ij}$. □

Lemma 4.1.3.

$$\begin{aligned}\partial_t \int Re^{-f} dV = &\int (H(\Delta f - |\nabla f|^2) + h_{pq}(-\nabla_p \nabla_q f + \nabla_p \nabla_q f) - R_{ij} h_{ij})e^{-f} dV \\ &+ \int (\frac{H}{2} - s)e^{-f} dV.\end{aligned}$$

Proof.

$$\begin{aligned}\partial_t \int Re^{-f}dV &= \int (\partial_t R)e^{-f}dV + \int R\partial_t(e^{-f}dV)\\ &= \int (-\Delta H - \nabla_i\nabla_j h_{ij} - R_{ij}h_{ij})e^{-f}dV + \int (\frac{H}{2} - s)e^{-f}dV\\ &= \int (H(\Delta f - |\nabla f|^2) + h_{pq}(-\nabla_p\nabla_q f + \nabla_p\nabla_q f) - R_{ij}h_{ij})e^{-f}dV\\ &\quad + \int (\frac{H}{2} - s)e^{-f}dV,\end{aligned}$$

where we use the standard evolution equation for R; i.e. $\partial_t R = -\Delta H - \nabla_i\nabla_j h_{ij} - R_{ij}h_{ij}$ and integration by parts. □

Lemma 4.1.4.

$$\partial_t \int |\nabla f|^2 e^{-f}dV = \int (-h_{ij}\nabla_i f\nabla_j f - 2s(\Delta f - |\nabla f|^2))e^{-f}dV + \int (\frac{H}{2} - s)e^{-f}dV.$$

Proof.

$$\begin{aligned}\partial_t \int |\nabla f|^2 e^{-f}dV &= \int (\partial_t|\nabla f|^2)e^{-f}dV + \int |\nabla f|^2\partial_t(e^{-f}dV)\\ &= \int (-h_{ij}\nabla_i f\nabla_j f + 2\langle\nabla_i s, \nabla_i f\rangle)e^{-f}dV\\ &\quad + \int |\nabla f|^2(\frac{H}{2} - s)e^{-f}dV\\ &= \int (-h_{ij}\nabla_i f\nabla_j f - 2s(\Delta f - |\nabla f|^2))e^{-f}dV\\ &\quad + \int |\nabla f|^2(\frac{H}{2} - s)e^{-f}dV,\end{aligned}$$

where we use integration by parts in the third equality. □

Lemma 4.1.5.

$$\partial_t \int |F|^2 e^{-f}dV = 2\int (-h_{ij}F_{ik}F_{jk} + b_i(d^*F_i + \iota_{\nabla f}da))e^{-f}dV + \int (\frac{H}{2} - s)e^{-f}dV.$$

Proof.

$$\begin{aligned}
\partial_t \int |F|^2 e^{-f} dV &= \int (\partial_t |F|^2) e^{-f} dV + \int |F|^2 \partial_t (e^{-f} dV) \\
&= \int \partial_t g^{ik} g^{jl} F_{ik} F_{jl} e^{-f} dV \\
&\quad + \int |F|^2 (\frac{H}{2} - s) e^{-f} dV \\
&= 2 \int (-h_{ij} F_{ik} F_{jk} + (db, F)) e^{-f} dV \\
&\quad + \int |F|^2 (\frac{H}{2} - s) e^{-f} dV \\
&= 2 \int (-h_{ij} F_{ik} F_{jk}) e^{-f} dV + 2 \int (db, e^{-f} F) dV \\
&\quad + \int |F|^2 (\frac{H}{2} - s) e^{-f} dV
\end{aligned}$$

$$\begin{aligned}
&= 2 \int (-h_{ij} F_{ik} F_{jk}) e^{-f} dV + 2 \int (b, d^*(e^{-f} F)) dV \\
&\quad + \int |F|^2 (\frac{H}{2} - s) e^{-f} dV,
\end{aligned}$$

where in the last line we used the definition of d^*. Now we also know that $d^*\alpha_i = -\nabla^j \alpha_{ji}$ for a 2-form. Thus

$$d^*(e^{-f} F) = -g^{jk} \nabla_k (e^{-f} F_{ji}) = e^{-f} d^* F + e^{-f} \nabla^j f F_{ji}.$$

But $F_{ij} = da_{ij}$, so this last term is $e^{-f} \iota_{\nabla f} da$. □

By combining the lemmas, we have that

$$\begin{aligned}\partial_t \mathcal{F}[g,a,f](h,b,s) \;&=\; \int (h_{ij}(-R_{ij} - \nabla_i\nabla_j f + \frac{1}{2}F_{ik}F_{jk}) \\ &+ b_i(-\frac{1}{2}d^* F_i - \frac{1}{2}\iota_{\nabla f} da_i))e^{-f}dV \\ &+ \int (\frac{trh}{2} - s)(2\Delta f - |\nabla f|^2 + R + |F|^2)e^{-f}dV.\end{aligned}$$

Following Perelman, we define $s = \frac{trh}{2}$, so that the volume $e^{-f}dV$ is fixed. We define a metric on our configuration space to be

$$\langle (g_1, a_1), (g_2, a_2)\rangle = \int (2(g_1, g_2) + 2(a_1, a_2))e^{-f}dV.$$

Then the gradient flow of $\mathcal{F}$ becomes

$$\partial_t g_{ij} = -2R_{ij} - 2\nabla_i\nabla_j f + F_{ik}F_{jk} \tag{4.1a}$$

$$\partial_t a_i = -d^* F_i - \iota_{\nabla f} da_i \tag{4.1b}$$

$$\partial_t f = -\Delta f - R + \frac{1}{2}|F|^2. \tag{4.1c}$$

Under equations (4.1), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(g(t), a(t), f(t)) =& 2\int |R_{ij} - \frac{1}{2}F_i^k F_{kj} + \nabla_i\nabla_j f|^2 e^{-f}dV \\ &+ \frac{1}{2}\int |d^\star F + \nabla_j f F_{ji}|^2 e^{-f}dV.\end{aligned}$$

Theorem 4.1.6. *Under equations (4.1),*

$$\frac{d}{dt}\mathcal{F}(g(t), a(t), f(t)) \geq 0. \tag{4.2}$$

Equality is a attained precisely when $R_{ij} - \frac{1}{2}F_i^k F_{kj} + \nabla_i\nabla_j f = 0$ *and* $d^\star F_j + \nabla_i f F_{ij} = 0$*; i.e. when* (g, a) *is a steady gradient Ricci Yang-Mills soliton.*

Notice that this corresponds to a monotonicity formula for the Ricci Yang-Mills flow. In fact, solutions to equations (4.1) are equivalent to the system

$$\partial_t g_{ij} = -2R_{ij} + F_{ik}F_{jk} \tag{4.3a}$$

$$\partial_t a_i = -d^* F_i \tag{4.3b}$$

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{1}{2}|F|^2, \tag{4.3c}$$

simply by pulling back by diffeomorphisms.

Corollary 4.1.7. *Under equations (4.3),*

$$\frac{d}{dt}\mathcal{F}(g(t), a(t), f(t)) \geq 0.$$

Proposition 4.1.8. *There exists a unique minimizer $\bar{f}$ of $\mathcal{F}(g, a, f)$ subject to the constraint $\int e^{-f} dV = 1$.*

Proof. We follow the proof of Kleiner and Lott [16]. Notice that $|\nabla e^{-\frac{f}{2}}|^2 = |-\frac{\nabla f}{2}e^{-\frac{f}{2}}|^2 = \frac{1}{4}|\nabla f|^2 e^{-f}$. Then

$$\begin{aligned}\mathcal{F} &= \int (|\nabla f|^2 + R - \frac{1}{4}|F|^2)e^{-f} dV \\ &= \int (4|\nabla e^{-\frac{f}{2}}|^2 + (R - \frac{1}{4}|F|^2)e^{-f}) dV.\end{aligned}$$

Let $\Phi = e^{-\frac{f}{2}}$, so that the constraint equation becomes $\int \Phi^2 dV = 1$. Then

$$\begin{aligned}\mathcal{F} &= \int (4|\nabla\Phi|^2 + (R - \frac{1}{4}|F|^2)\Phi^2) dV \\ &= \int \Phi(-4\Delta\Phi + (R - \frac{1}{4}|F|^2)\Phi) dV.\end{aligned}$$

Thus Φ is the smallest eigenfunction of $-4\Delta + R - \frac{1}{4}|F|^2$. $\square$

We can then define $\lambda(g,a) = \inf\{\mathcal{F}(g,a,f) : f \in \mathcal{C}^\infty, \int e^{-f} dV = 1\}$, and we claim that λ is also a monotonic quantity.

Proposition 4.1.9. *If $(g(\cdot), a(\cdot))$ is a solution to the Ricci Yang-Mills flow, then $\lambda(g(t), a(t))$ is non-decreasing in time.*

Proof. We follow the proof in [6]. Let $t_0 \in [0, T]$ and f_0 be the unique minimizer of $\mathcal{F}(g(t), a(t), f(t))$ at t_0. Then $\lambda(g(t_0), a(t_0)) = \mathcal{F}(g(t_0), a(t_0), f_0)$. Let f solve the backwards heat equation

$$\frac{\partial}{\partial t} f = -\Delta f - R + \frac{1}{2}|F|^2 + |\nabla f|^2$$
$$f(t_0) = f_0$$

on the interval $[0, t_0]$. We know that $\frac{d}{dt}\mathcal{F} \geq 0$ for $t \leq t_0$. Also the backwards heat equation preserves the constraint $\int e^{-f} dV = 1$. So $\lambda(g(t), a(t)) \leq \mathcal{F}(g(t), a(t), f(t))$ for $t \leq t_0$. Thus

$$\lambda(g(t), a(t)) \leq \mathcal{F}(g(t), a(t), f(t)) \leq \mathcal{F}(g(t_0), a(t_0), f(t_0)) = \lambda(g(t_0), a(t_0)).$$

So λ is non-decreasing. □

Notice that the minimum value of $\lambda(g,a)$ is equal to $\lambda_1(g,a)$, where $\lambda_1(g,a)$ is the smallest eigenvalue of the elliptic operator $-4\Delta + R - \frac{1}{4}|F|^2$. Then the minimizer, f_0, of $\mathcal{F}$ satisfies the Euler-Lagrange equation

$$\lambda(g,a) = 2\Delta f_0 - |\nabla f_0|^2 + R - \frac{1}{4}|F|^2. \tag{4.4}$$

4.2 Breathers

Definition 4.2.1. A solution $(g(t), a(t))$ to the Ricci Yang-Mills flow is called a *breather* if there exist times $t_1 < t_2$, a constant α, and a diffeomorphism $\phi : M \to M$ such that

$$g(t_2) = \alpha\phi^* g(t_1),\ \ a(t_2) = \alpha\phi^* a(t_1).$$

$\alpha > 1$, $\alpha < 1$, and $\alpha = 1$ correspond to $(g(t), a(t))$ being a *expanding*, *shrinking*, or *steady* breather.

Lemma 4.2.1. (No nontrivial steady breathers) *Let $(M^n, g(t), a(t))$ be a solution to the Ricci Yang-Mills flow on a closed manifold. If there exist $t_1 < t_2$ with $\lambda(g(t_1), a(t_1)) = \lambda(g(t_2), a(t_2))$ (i.e. the solution is a steady breather), then $(g(t), a(t))$ is a steady gradient Ricci Yang-Mills soliton, which must have $|F|^2 = 0$ and be scalar flat.*

Proof. Suppose there exist $t_1 < t_2$ with $\lambda(g(t_1), a(t_1)) = \lambda(g(t_2), a(t_2))$. Then let f_2 be the minimizer for $\mathcal{F}$ at t_2, so that $\mathcal{F}(g(t_2), a(t_2), f_2) = \lambda(g(t_2), a(t_2))$. Let f be the solution to the usual backwards heat equation with $f(t_2) = f_2$. Then

$$\begin{aligned}
\lambda(g(t_1), a(t_1)) &\leq \mathcal{F}(g(t_1), a(t_1), f(t_1)) \\
&\leq \mathcal{F}(g(t), a(t), f(t)) \\
&\leq \mathcal{F}(g(t_2), a(t_2), f(t_2)) \\
&= \lambda(g(t_2), a(t_2)),
\end{aligned}$$

for $t \in [t_1, t_2]$. Since $\lambda(g(t_1), a(t_1)) = \lambda(g(t_2), a(t_2))$ and λ monotonic, we have that

$$\mathcal{F}(g(t), a(t), f(t)) = \lambda(g(t), a(t)) \equiv \text{constant on } [t_1, t_2].$$

Thus $2 \int |R_{ij} - \frac{1}{2} F_i^k F_{kj} + \nabla_i \nabla_j f|^2 e^{-f} dV + \frac{1}{2} \int |d^\star F + \nabla_j f F_{ji}|^2 e^{-f} dV = 0$, so

$$R_{ij} - \frac{1}{2} F_i^k F_{kj} + \nabla_i \nabla_j f = 0 \tag{4.5a}$$

$$d^\star F + \nabla_j f F_{ji} = 0. \tag{4.5b}$$

In other words, (g, a) is a steady Ricci Yang-Mills soliton. Consider equation (4.5b).

Using integration by parts, we see that

$$\begin{aligned} 0 &= \int (d^\star F_j + \nabla^i f F_{ij}, a_j) e^{-f} dV \\ &= \int ((F_{ij}, F_{ji}) - (F_{ij}, a_j \nabla_i f) + (\nabla^i f F_{ij}, a_j)) e^{-f} dV \\ &= -\int |F|^2 e^{-f} dV. \end{aligned}$$

So we must have that $|F|^2 = 0$. Since f is a minimizer of λ, f satisfies equation (4.4): $2\Delta f - |\nabla f|^2 + R = \lambda_1(g(t), a(t)) = \lambda(g(t), a(t))$. We can take the trace of equation (4.5a) to obtain $R + \Delta f = 0$. Then $R + |\nabla f|^2 = -\lambda$. Integrating, we have

$$-\lambda = \int (R + |\nabla f|^2) e^{-f} dV = \lambda,$$

so $\lambda = 0$ and $\Delta f = |\nabla f|^2 = -R$. Then

$$0 = \int (\Delta f - |\nabla f|^2) e^f dV = -2 \int |\nabla f|^2 e^f dV,$$

so f is constant. Thus equation (4.5a) implies that $R = 0$. $\square$

Chapter 5

Stability

We would like to use maximal regularity theory to study the stability of the Ricci Yang-Mills flow at a fixed point.

5.1 Motivation

The theory that we will use to study stability of the Ricci Yang-Mills flow is suitably technical, but intuitively, it can be understood through some basic ODE examples.

Example 1. The linear system

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \end{cases}$$

has $(0,0)$ as a fixed point. This fixed point is stable, in that any solution that gets "close enough" to $(0,0)$ will flow towards it. The ODE system can be written in the form $\dot{\mathbf{x}} = J\mathbf{x}$, where $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The negative eigenvalues of J correspond to the stability of the fixed point.

Example 2. The linear system

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = 0 \end{cases}$$

has every point on the x_2-axis as a fixed point. Every fixed point is unstable; i.e. slight perturbations cause solutions to move away from the fixed point. In this case $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so the zero eigenvalue corresponds to a line of fixed points, while the positive eigenvalue causes the instability.

Example 3. The nonlinear system

$$\begin{cases} \dot{x}_1 = -x_1 - x_2^3 \\ \dot{x}_2 = -x_2 + x_1^3 \end{cases}$$

has $(0,0)$ as a fixed point. To study the behavior of this system near $(0,0)$, we can use the principle of linear stability to claim that the local phase plane portrait is the same as that of Example 1.

Example 4. Suppose we are given an ODE in Euclidean space of the form

$$\begin{cases} \dot{x}(t) = ax(t) + f(t) \\ \quad x(0) = x_0. \end{cases}$$

Assuming suitable conditions are satisfied, the variation of constants formula implies that solutions are of the form

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-s)} f(s)ds.$$

We will consider the Ricci Yang-Mills flow to be an ODE on an infinite dimensional space. We will linearize the right-hand side of the equation and study the spectrum of that operator. This will determine the local behavior of the flow.

Consider the general equation

$$\frac{\partial u}{\partial t} = \Phi(x, t, u, Du, D^2u) \tag{5.1a}$$

$$u(0) = u_0. \tag{5.1b}$$

For our purposes, this will be a (weakly) parabolic system of PDE. Let $\bar{u}$ be a fixed point of the equation (i.e. $\Phi(\bar{u}) = 0$), and let $\Sigma = \sigma(D_{\bar{u}}\Phi) \cap \mathbb{R}$. Then

Definition 5.1.1. $\bar{u}$ is

- *linearly stable* if $\Sigma \subset (-\infty, 0]$,
- *strictly linearly stable* if $\Sigma \subset (-\infty, 0)$,
- *linearly unstable* if $\Sigma \cap (0, \infty) \neq \emptyset$,
- *asymptotically stable* if there exists a neighborhood about $\bar{u}$ such that every solution of the equation having initial data in that neighborhood exists for all positive time and converges to $\bar{u}$ as $t \to \infty$.

5.2 Asymptotic Behavior of Quasilinear Partial Differential Equations

In the case of the Ricci Yang-Mills flow over a compact surface, we will see that the linearized operator of the right hand side, in the notation above, either has a zero eigenvalue or has a strictly negative spectrum. Analogous to Example 2, a zero eigenvalue will correspond to the existence of a finite dimensional center manifold, while negative eigenvalues will correspond to a stable fixed point. We will use the same general analysis in both cases and will point out the differences in the techniques.

5.2.1 Center Manifold Theorem

We would like to analyze the stability of autonomous quasilinear parabolic equations. Suppose that Φ in equation (5.1a) is a quasilinear elliptic operator, and suppose that we are in the case where $\sup\{\Re(\lambda) : \lambda \in \sigma(D_{\bar{u}}\Phi)\} \geq 0$. This critical case is complicated, so it is only treatable if we work with interpolation spaces. We will consider the case that

$$\sigma_+(D_{\bar{u}}\Phi) := \{\lambda \in \sigma(D_{\bar{u}}\Phi) : \Re(\lambda) \geq 0\}$$

consists of a finite number of isolated eigenvalues with finite algebraic multiplicity.

In [9], Guenther, Isenberg, and Knopf used the general theory of [24] to study the stability of the Ricci flow at Ricci-flat metrics. Additionally, Knopf has recently used these techniques to study the stability of locally $\mathbb{R}^N$-invariant solutions of Ricci flow [17]. We would like to study the stability of Ricci Yang-Mills flow at Einstein Yang-Mills metrics using [24], which essentially shows that if Φ is a quasilinear differential operator satisfying certain conditions, with $D\Phi$ having a zero eigenvalue, then the local behavior of the flow near a fixed point is characterized by the presence of a local center manifold. We give a more detailed look at some of the ideas in [24] in the appendix. The theorem that we will use is based upon Theorem 2.2 in [9], which in turn is a compendium of results from Theorems 4.1 and 5.8 and Remark 4.2 in [24]. The set-up for the theorem is somewhat complicated, so we first collect the necessary assumptions.

Center Manifold Theorem Requirements:

Let $\mathbb{X}_1 \hookrightarrow \mathbb{X}_0$ be a continuous dense inclusion of Banach spaces, and let $\mathbb{X}_\alpha$ and $\mathbb{X}_\beta$ denote the continuous interpolation spaces corresponding to fixed $0 < \beta < \alpha < 1$. In other words, $\mathbb{X}_\alpha = (X_0, X_1)_\alpha$ and similarly for $\mathbb{X}_\beta$. Let

$$\frac{\partial}{\partial t}\vec{x} = A(\vec{x})\vec{x} \tag{5.2}$$

be an autonomous quasilinear parabolic equation posed for $t \geq 0$. Suppose that $\mathcal{U}_\beta \subset \mathbb{X}_\beta$ is an open set and that

$$A(\cdot) \in C^k(\mathcal{U}_\beta, L(\mathbb{X}_1, \mathbb{X}_0))$$

for some positive integer k.

Additionally, assume that there exists a pair $\mathbb{E}_1 \hookrightarrow \mathbb{E}_0$ of Banach spaces and that there exists an extension $\tilde{A}(\cdot)$ of $A(\cdot)$ to domain $D(\tilde{A}(\cdot))$ that is dense in $\mathbb{E}_0$. We would like the following statements to hold for all $\vec{x} \in \mathcal{U}_\alpha = \mathcal{U}_\beta \cap \mathbb{X}_\alpha$:

(a). $\tilde{A}(\vec{x}) \in L(\mathbb{E}_1, \mathbb{E}_0)$ generates a strongly continuous analytic semigroup on $L(\mathbb{E}_0)$;

(b). $\mathbb{X}_0 \cong D_{\tilde{A}(\vec{x})}(\theta) \cong (\mathbb{E}_0, D(\tilde{A}(\vec{x})))_\theta$ and $\mathbb{X}_1 \cong D_{\tilde{A}(\vec{x})}(1+\theta) \cong (\mathbb{E}_0, D(\tilde{A}(\vec{x})))_{1+\theta}$ for some $\theta \in (0,1)$. Here $(\cdot,\cdot)$ denotes the continuous interpolation method. Also $D_{\tilde{A}(\vec{x})}(1+\theta) = \{\vec{x} \in D(\tilde{A}) : \tilde{A}\vec{x} \in D_{\tilde{A}(\vec{x})}(\theta)\}$.

(c). $A(\vec{x})$ agrees with the restriction of $\tilde{A}(\vec{x})$ to the dense subset $D(A) \subseteq \mathbb{X}_0$;

(d). $\mathbb{E}_1 \hookrightarrow \mathbb{X}_\beta \hookrightarrow \mathbb{E}_0$ is a continuous and dense inclusion with the property that there are $C > 0$ and $\delta \in (0,1)$ such that for all $\eta \in \mathbb{E}_1$, one has

$$||\eta||_{\mathbb{X}_\beta} \leq C||\eta||_{\mathbb{E}_0}^{1-\delta}||\eta||_{\mathbb{E}_1}^{\delta}.$$

Let $\hat{x} \in \mathfrak{U}_\alpha$ be a fixed point of equation (5.2). Suppose the spectrum σ of the linearized operator $DA|_{\hat{x}}$ admits the decomposition $\sigma = \sigma_s \cup \sigma_{cu}$, where $\sigma_s \subset \{z : \Re(z) < 0\}$ and where $\sigma_{cu} \subset \{z : \Re(z) \geq 0\}$ consists of finitely many eigenvalues of finite multiplicity. Suppose further that $\sigma_{cu} \cap i\mathbb{R} \neq \emptyset$.

Theorem 5.2.1. (Statement of the Center Manifold Theorem)

1. *If $S(\lambda)$ denotes the algebraic eigenspace of $\lambda \in \sigma_{cu}$, then $\mathbb{X}_\alpha$ admits the decomposition $\mathbb{X}_\alpha = \mathbb{X}_\alpha^s \oplus \mathbb{X}_\alpha^{cu}$ for all $\alpha \in [0,1]$, where $\mathbb{X}_\alpha^{cu} \equiv \oplus_{\lambda \in \sigma_{cu}} S(\lambda)$.*

2. *For each $r \in \mathbb{N}$, there exists $d_r > 0$ such that for all $d \in (0, d_r]$, there is a C^r manifold $\mathcal{M}_{loc}^{cu}$ that is locally invariant for solutions of (5.2) as long as they remain in $B(\mathbb{X}_1^{cu}, \hat{x}, d) \times B(\mathbb{X}_1^s, 0, d)$. It satisfies $T_{\hat{x}}\mathcal{M}_{loc}^{cu} \cong \mathbb{X}_1^{cu}$, so that $\mathcal{M}_{loc}^{cu}$ is a* local center manifold *if $\sigma_{cu} \subset i\mathbb{R}$ and a* local center unstable manifold *otherwise.*

3. *For all $\alpha \in (0,1)$, there are constants $C_\alpha > 0$ independent of $\hat{x}$ and constants $\omega > 0$ and $\hat{d} \in (0, d_0]$ such that one has*

$$||\pi^s(\vec{x}(t)) - \phi(\pi^{cu}\vec{x}(t))||_{\mathbb{X}_1} \leq \frac{C_\alpha}{t^{1-\alpha}}e^{-\omega t}||\pi^s(\vec{x}(0)) - \phi(\pi^{cu}\vec{x}(0))||_{\mathbb{X}_\alpha}$$

   for all solutions $\vec{x}(t)$ with $\vec{x}(0) \in B(\mathbb{X}_\alpha, \hat{x}, d)$ and all times $t \geq 0$ such that the solution $\vec{x}(t)$ remains in $B(\mathbb{X}_\alpha, \hat{x}, d)$. Here π^s and π^{cu} denote the projections onto $\mathbb{X}_\alpha^s \cong (\mathbb{X}_1^s, \mathbb{X}_0^s)_\alpha$ and $\mathbb{X}_\alpha^{cu}$ respectively.

Proof. For the full proof of the theorem, we refer the reader to Sections 4 and 5 of [24]. In the appendix, we provide a sketch of the existence of the local center manifolds. □

5.2.2 Asymptotic Stability Theorem

Now suppose that we are in the case of $\sigma_{cu} = \emptyset$; i.e. $\sigma(A) \equiv \sigma_s$. In this case, our "center manifold" will consist of a single point. We would still like to use the machinery of Simonett, as this monopolizes upon the smoothing properties of quasilinear parabolic equations and yields an optimal regularity result that solutions in a $\mathbb{X}_\alpha$-neighborhood of a fixed point converge exponentially fast in $\mathbb{X}_1$-norm to the fixed point. We use the following adaptation of Theorem 5.2.1 to show that a fixed point is asymptotically stable.

Theorem 5.2.2. (Statement of the Asymptotic Stability Theorem) *Suppose that the hypotheses of Theorem 5.2.1 are satisfied. As before, let $\hat{x} \in \mathfrak{U}_\alpha$ be a fixed point of equation (5.2). Suppose also that* $\sup\{\Re(\lambda) : \lambda \in \sigma\} \leq -\delta$ *for some $\delta > 0$. Then for all $\alpha \in (0,1)$, there are constants $C_\alpha > 0$ independent of $\hat{x}$ and constants $\omega > 0$ and $\hat{d} \in (0, d_0]$ such that one has*

$$||\vec{x}(t) - \hat{x}||_{\mathbb{X}_1} \leq \frac{C_\alpha}{t^{1-\alpha}} e^{-\omega t} ||\vec{x}(0) - \hat{x}||_{\mathbb{X}_\alpha},$$

for all solutions $\vec{x}(t)$ with $\vec{x}(0) \in B(\mathbb{X}_\alpha, \hat{x}, d)$ and all times $t \geq 0$ such that the solution $\vec{x}(t)$ remains in $B(\mathbb{X}_\alpha, \hat{x}, d)$.

Remark 5.2.1. One can also use the theory of semigroups to prove a slightly less general result. In [18], Lunardi shows exponential convergence to a fixed point for quasilinear parabolic PDE for solutions having initial data in a $\mathbb{X}_1$-neighborhood of the fixed point. This analysis is slightly less complicated than that of Simonett and has the advantage that it can be extended to fully nonlinear PDE, as in Chapter 9 of [19].

5.3 Linearizing the Flow

We would like to consider the stability of our flow at a fixed point. On a surface, we can let $g = e^u h$, where h is a fixed constant curvature metric. The Ricci Yang-Mills flow equations then become

$$\partial_t u = \Delta_g u - R_h e^{-u} + \frac{1}{2}|F|^2 \tag{5.3a}$$

$$\partial_t a = -d^\star F. \tag{5.3b}$$

Notice that the equation for a is not quite parabolic; the RHS is comprised of "one-half" of the laplacian. We remedy this by using a 1-parameter family of diffeomorphisms.

Lemma 5.3.1. *Equation 5.3 is equivalent to a parabolic flow via pullback by diffeomorphisms.*

Proof. We make the same choice of vector field that we used in the short-time existence proof. Namely, let $W^k = g^{ij}(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})$ for $k = 1, 2$, where $\tilde{\Gamma}$ is the Christoffel symbol with respect to the fixed background metric h. Notice that if $g_{ij} = e^u h_{ij}$ on a surface, then $W^k = 0$. Additionally, let $W^3 = -d^\star a$. Then, if h is the metric on our principal bundle in the Kaluza-Klein ansatz satisfying equation (5.3), then $\phi_t^* h$ satisfies

$$\partial_t u = \Delta_g u - R_h e^{-u} + \frac{1}{2}|F|^2 \tag{5.4a}$$

$$\partial_t a = -d^\star F - dd^\star a, \tag{5.4b}$$

where ϕ_t is the one-parameter family of diffeomorphisms generated by W. We will call this flow GRYM, and we will choose to work with these equations as they are parabolic. □

We would like to do stability analysis of the fixed points of our flow. We would like these fixed points to be the natural geometric limit of the Ricci Yang-Mills flow; namely, we want them to be Einstein Yang-Mills metrics. An Einstein Yang-Mills metric is one such that g is Einstein and $d^\star F_a = 0$. In order to make this work, we must consider a normalized version of equations (5.3) which we will call NGRYM. We have

$$\partial_t u = \Delta_g u - R_h e^{-u} + \frac{1}{2}|F|^2 + r - \frac{1}{2}f \tag{5.5a}$$

$$\partial_t a = -d^* F - dd^* a, \tag{5.5b}$$

where r and f are the averages of scalar curvature and bundle curvature, respectively; i.e. $r = \frac{\int_M R dV}{\int dV}$ and $f = \frac{\int_M |F|^2 dV}{\int dV}$. Since M is a surface, r is a constant in space and time.

Remark 5.3.1. NGRYM should be thought of as a certain volume-normalizing flow, in that the volume of the base manifold is fixed. Due to the lack of scale invariance on the RHS of equation (5.4a), this flow is not quite a rescaling of our original equation. However, there is evidence ([26]) that a normalization of this form is useful in proving convergence of the flow. So our results will be applied to this closely related flow.

We claim that an EYM manifold is a fixed point of the flow. To see this, we need the following lemma.

Lemma 5.3.2. *In the case of a $U(1)$ bundle over a compact surface M, a Yang-Mills connection has the property that its curvature is a constant times the volume form; i.e. $F = \lambda dV$, where λ is determined by the Chern number of M.*

Proof. We can write F as $F = f(x)dV$ for some function f. F being the curvature of a Yang-Mills connection implies that $d^*F = 0$, and since $U(1)$ is abelian, $dF = 0$. Thus $(dd^* + d^*d)F = (dd^* + d^*d)(f(x)dV) = 0$. Since $d(dV) = d^*(dV) = 0$, the above shows that $\Delta f = 0$. As M is compact, f must be a constant λ.

By definition, the Chern number c of a $U(1)$ bundle is given by

$$c = \frac{1}{2\pi}\int_M F.$$

But now, we have $c = \frac{1}{2\pi}\int_M \lambda dV$, so $\lambda = \frac{2\pi c}{\int_M dV}$. □

Also for an Einstein metric, $u = C$ for some constant C. If we write our Yang-Mills connection in the Coulomb gauge ($d^*a = 0$), then we see that an EYM manifold is a fixed point of the NGRYM.

Theorem 5.3.3. *Let M be a surface of genus $g \geq 1$. The space of Yang-Mills $U(1)$-connections having fixed curvature λ over M modulo the gauge group has dimension $2g$.*

Proof. We begin in the case of $\lambda = 0$. Namely, it can be shown, as in [14], that

$$\mathcal{M} \cong \{\text{representations of } \pi \text{ into G}\}/\{\text{conjugation}\},$$

where $\mathcal{M} := \{\text{flat connections}\}/\{\text{gauge group}\}$ and π is the fundamental group of M. One can in fact define explicitly a homomorphism from the space of flat connections to the space of representations of π into G. According to [1], the same holds for non-zero Yang-Mills connections. Then for a genus $g \geq 1$ surface with $G = U(1)$, $\mathcal{M} = U(1)^{2g}$. So the dimension of $\mathcal{M}$ is $2g$. □

We can compute the linearization about an EYM manifold of the RHS of equation (5.5) in the standard fashion. Let $\frac{\partial u}{\partial t}\big|_{t=0} = v$ and $\frac{\partial a}{\partial t}\big|_{t=0} = b$. Notice that this implies that $\frac{\partial F}{\partial t}\big|_{t=0} = db$. Then we have the following computations.

Lemma 5.3.4.

$$\frac{\partial}{\partial t}\bigg|_{t=0} (\Gamma^k_{ij}) = \frac{1}{2}(\partial_i v \delta^k_j + \partial_j v \delta^k_i - \partial^k v h_{ij}). \tag{5.6}$$

Proof. Notice that if $g = e^u h$, then the following holds.

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \\
&= \frac{1}{2} e^{-u} h^{kl}(\partial_i(e^u h_{jl}) + \partial_j(e^u h_{il}) - \partial_l(e^u h_{ij})) \\
&= \frac{1}{2} g^{kl}((\partial_i u g_{jl} + \partial_j u g_{il} - \partial_l u g_{ij}) + \Gamma(h)_{ij}^k \\
&= \frac{1}{2}(\partial_i u \delta_j^k + \partial_j u \delta_i^k - \partial^k u h_{ij}) + \Gamma(h)_{ij}^k.
\end{aligned}$$

Using this identity, we obtain:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\Gamma_{ij}^k) = \frac{1}{2}(\partial_i v \delta_j^k + \partial_j v \delta_i^k - \partial^k v h_{ij}).$$

□

Lemma 5.3.5.

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (\Delta u) = \Delta v - v\Delta u. \tag{5.7}$$

Proof.

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0} (\Delta_g u) &= \left.\frac{\partial}{\partial t}\right|_{t=0} (g^{ij}\nabla_i \nabla_j u) \\
&= -v\Delta u + \Delta v - g^{ij}(\left.\frac{\partial}{\partial t}\right|_{t=0} (\Gamma_{ij}^m)\partial_m u) \\
&= -v\Delta u + \Delta v - g^{ij}(\frac{1}{2}(\partial_i v \delta_j^m + \partial_j v \delta_i^m - \partial^m v g_{ij})\partial_m u) \\
&= \Delta v - v\Delta u.
\end{aligned}$$

□

Lemma 5.3.6.

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (e^{-u}) = -ve^{-u}. \tag{5.8}$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}\bigg|_{t=0} (e^{-u}) &= -\frac{\partial}{\partial t}\bigg|_{t=0} (u)e^{-u} \\ &= -ve^{-u}. \end{aligned}$$

□

Lemma 5.3.7.

$$\frac{\partial}{\partial t}\bigg|_{t=0} (|F|^2) = -2v|F|^2 + 2\langle db, F\rangle. \tag{5.9}$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}\bigg|_{t=0} (|F|^2) &= \frac{\partial}{\partial t}\bigg|_{t=0} (g^{ij}g^{kl}F_{ik}F_{jl}) \\ &= 2(-v|F|^2) + 2\langle db, F\rangle. \end{aligned}$$

□

Lemma 5.3.8.

$$\frac{\partial}{\partial t}\bigg|_{t=0} (-d^*F_j) = -d^*db_j + vd^*F_j - \nabla^i vF_{ij}. \tag{5.10}$$

Proof.

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0}(-d^*F_j) &= \left.\frac{\partial}{\partial t}\right|_{t=0}\nabla^i F_{ij} \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0}(g^{ki}(\partial_k da_{ij} - \Gamma^m_{ki}da_{mj} - \Gamma^m_{kj}da_{im}) \\
&= -v\nabla^i da_{ij} + \nabla^i db_{ij} - \left.\frac{\partial}{\partial t}\right|_{t=0}\Gamma^m_{ki}da_{mi} - \left.\frac{\partial}{\partial t}\right|_{t=0}\Gamma^m_{kj}da_{im} \\
&= -v\nabla^i da_{ij} + \nabla^i db_{ij} - \frac{1}{2}(\partial^i vF_{ij} + \partial^k vF_{kj} - 2\partial^m vF_{mj}) \\
&\quad -\frac{1}{2}(\partial^i vF_{ij} - \partial^m vF_{im}) \\
&= \nabla^i db_{ij} - v\nabla^i da_{ij} - \nabla^i vF_{ij} \\
&= -d^*db_j + vd^*F_j - \nabla^i vF_{ij}.
\end{aligned}$$

□

Lemma 5.3.9.

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(d^*a) = d^*b + v\nabla^j a_j + \partial^k uva_k. \tag{5.11}$$

Proof.

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_{t=0}(d^*a) &= \left.\frac{\partial}{\partial t}\right|_{t=0}(-g^{ij}\nabla_i a_j) \\
&= v\nabla^j a_j - \nabla^i b_j + g^{ij}\left.\frac{\partial}{\partial t}\right|_{t=0}(\Gamma^k_{ij})a_k \\
&= v\nabla^j a_j - \nabla^i b_j + \frac{1}{2}(\partial^i va_i + \partial^i va_i - 2\partial^k va_k) + \partial^k uva_k.
\end{aligned}$$

□

Now we would like the linearization of our equations at an EYM metric. We use the previous characterization of EYM manifolds, as well as the fact that a Yang-Mills connection is a minimizer of the Yang-Mills functional $\int |F|^2$. Let $L_1(v, b)$ denote the linearization of equation (5.5a) in the direction of (v, b) and $L_2(v, b)$ denote that of equation (5.5b).

Lemma 5.3.10. *The linearization of the right-hand side of equation (5.5) at an Einstein Yang-Mills metric is*

$$L_1(v, b) = \Delta v + (R_h - \lambda^2)v + \lambda\langle db, dV_h\rangle, \tag{5.12a}$$

$$L_2(v, b) = \Delta_d b - \lambda\nabla^i v dV_{ij}. \tag{5.12b}$$

Proof. By the above lemmata, we see that the linearization of the RHS of equation (5.5a) is

$$L_1(v, b) = \Delta v - v\Delta u + vR_h e^{-u} - v|F|^2 + \langle db, F\rangle.$$

Notice that at a EYM metric, $|F|^2 = |\lambda dV|^2 = \lambda^2$, since we are in complex dimension 1. Then we have

$$L_1(v, b) = \Delta v + (R_h - \lambda^2)v + \lambda\langle db, dV\rangle.$$

Similarly, the linearization of the RHS of equation (5.5b) is

$$L_2(v, b)_j = -d^* db_j - dd^* b + vd^* F_j - \nabla^i v F_{ij} + d(vd^* a)_j + d(\partial^k uva_k)_j.$$

Then using the fact that both $d^* F = 0$ and $d^* a = 0$, as well as the above characterizations, the result becomes

$$L_2(v, b)_j = \Delta_d b_j - \lambda\nabla^i v dV_{ij}.$$

□

5.4 Application of the Center Manifold Theorem to the NGRYM

We would like to show that we can apply Theorems 5.2.1 and 5.2.1 to NGRYM. In order to put our analysis into this framework, we need to define appropriate spaces.

Fix $0 < \delta < \epsilon < 1$. We would like the consider the following hierarchy of Banach spaces

$$\mathbb{X}_1 \subset \mathbb{E}_1 \subset \mathbb{X}_0 \subset \mathbb{E}_0,$$

where

$$\begin{aligned} \mathbb{X}_1 &= \{(u,a) : u \in \mathfrak{h}^{2+\epsilon}, a \in \mathfrak{h}^{2+\epsilon}\}, \\ \mathbb{E}_1 &= \{(u,a) : u \in \mathfrak{h}^{2+\delta}, a \in \mathfrak{h}^{2+\delta}\}, \\ \mathbb{X}_0 &= \{(u,a) : u \in \mathfrak{h}^{\epsilon}, a \in \mathfrak{h}^{\epsilon}\}, \\ \mathbb{E}_0 &= \{(u,a) : u \in \mathfrak{h}^{\delta}, a \in \mathfrak{h}^{\delta}\}. \end{aligned}$$

Notice that in the notation above, we mean $(u,a) \in \mathfrak{h}^{\cdot}(M) \times \mathfrak{h}^{\cdot}(\Omega^1(M))$. Here $\mathfrak{h}^{\cdot}$ is the little Hölder space defined in Appendix D. We would like to check that assumption (b) holds. That is, we want to show that $X_0 = (E_0, E_1)_\theta$ and $X_1 = (E_0, E_1)_{1+\theta}$. Notice that if $\theta = \frac{\epsilon-\delta}{2}$, then $\mathfrak{h}^{2+\epsilon} = (\mathfrak{h}^{2+\delta}, \mathfrak{h}^{\delta})_{1+\theta}$ and $\mathfrak{h}^{\epsilon} = (\mathfrak{h}^{2+\delta}, \mathfrak{h}^{\delta})_\theta$ by Lemma D.0.10. Then we use the following lemma to show that the product of interpolation spaces are also interpolation spaces.

Lemma 5.4.1. *If* $\mathbb{M}_\alpha \in J_\alpha(\mathbb{M}_0, \mathbb{M}_1)$ *and* $\mathbb{N}_\alpha \in J_\alpha(\mathbb{N}_0, \mathbb{N}_1)$ *then* $\mathbb{M}_\alpha \times \mathbb{N}_\alpha \in J_\alpha(\mathbb{M}_0 \times \mathbb{N}_0, \mathbb{M}_1 \times \mathbb{N}_0)$.

Proof. We would like to check the interpolation inequality, namely that for $(x,y) \in \mathbb{M}_0 \times \mathbb{N}_0$,

$$||(x,y)||_\alpha \leq c||(x,y)||_0^\alpha ||(x,y)||_1^{1-\alpha}, \tag{5.13}$$

where $||(x,y)||_a^2 := ||x||_a^2 + ||y||_a^2$. Consider the following:

$$\begin{aligned} ||(x,y)||_\alpha^2 &= ||x||_\alpha^2 + ||y||_\alpha^2 \\ &\leq c||x||_0^{2\alpha}||x||_1^{2(1-\alpha)} + c||y||_0^{2\alpha}||y||_1^{2(1-\alpha)} \\ &\leq c(||x||_0^2 + ||y||_0^2)^\alpha (||x||_1^2 + ||y||_1^2)^{1-\alpha} \\ &= c||(x,y)||_0^{2\alpha}||(x,y)||_1^{2(1-\alpha)}, \end{aligned}$$

where the third line follows from Hölder's inequality. □

Let $\mathbb{X}_\beta = (\mathbb{X}_0, \mathbb{X}_1)_\beta$ and $\mathbb{X}_\alpha = (\mathbb{X}_0, \mathbb{X}_1)_\alpha$ for fixed $\frac{1}{2} \leq \beta < \alpha < 1$. We would like to check that $\mathbb{E}_1 \hookrightarrow \mathbb{X}_\beta \hookrightarrow \mathbb{E}_0$ is a continuous and dense inclusion and that the interpolation inequality, equation (5.13), holds. This will fulfill assumption (d) of Theorem 5.2.1. Notice that X_0 and X_1 can be gotten from interpolating between E_0 and E_1. If $\theta = \frac{\epsilon - \delta}{2}$, then by lemma D.0.10,

$$\mathbb{X}_0 = (\mathbb{E}_0, \mathbb{E}_1)_\theta, \tag{5.14}$$

$$\mathbb{X}_1 = (\mathbb{E}_0, \mathbb{E}_1)_{1+\theta}. \tag{5.15}$$

Lemma 5.4.2. *There exists $C > 0$, $\rho \in (0,1)$, and $\beta \in (\frac{1}{2}, 1)$ such that $\mathbb{E}_1 \hookrightarrow \mathbb{X}_\beta \hookrightarrow \mathbb{E}_0$ and*

$$||\eta||_{\mathbb{X}_\beta} \leq C||\eta||_{\mathbb{E}_0}^{1-\rho}||\eta||_{\mathbb{E}_1}^\rho,$$

for all $\eta \in \mathbb{X}_\beta$.

Proof. Since $\beta \in (\frac{1}{2}, 1)$, it is clear that the inclusions hold. Using equations

(5.14) and (5.15),we have

$$\begin{aligned} ||\eta||_{\mathbb{X}_\beta} &\leq C||\eta||_{\mathbb{X}_0}^{1-\beta}||\eta||_{\mathbb{X}_1}^{\beta} \\ &\leq C(||\eta||_{\mathbb{E}_0}^{1-\theta}||\eta||_{\mathbb{E}_1}^{\theta})^{1-\beta}(||\eta||_{\mathbb{E}_0}^{1-(1+\theta)}||\eta||_{\mathbb{E}_1}^{1+\theta})^{\beta} \\ &\leq C||\eta||_{\mathbb{E}_0}^{1-\rho}||\eta||_{\mathbb{E}_1}^{\rho}, \end{aligned}$$

where $\rho = \theta + \beta$. It remains only to check that $\rho \in (0,1)$. Since $0 < \delta < \epsilon < 1$, $0 < \frac{\epsilon-\delta}{2} < \frac{1}{2}$, we simply choose β to be in $(\frac{1}{2}, 1 - \frac{\epsilon-\delta}{2})$. □

Since we would like to use Theorem 5.2.1, we want to make our notation match that of Simonett. In other words, we would like to write equation (5.5) as

$$\partial_t \vec{x} = A(\vec{x})\vec{x}.$$

In a fixed coordinate system, we can write the right-hand side of equation (5.5) as

$$A(u,a)(u,a) = \begin{pmatrix} a(x,g)^{ij}\partial_i\partial_j u + b(x,g,\partial g)^k\partial_k u + c(x,g,da)u \\ d(x,g)^{ij}\partial_i\partial_j a_k + e(x,g,\partial g)^i\partial_i a_k - f(x)a_k \end{pmatrix}, \quad (5.16)$$

where the functions $a(x,\cdot), b(x,\cdot,\cdot), c(x,\cdot,\cdot), d(x,\cdot), e(x,\cdot,\cdot), f(x)$ depend smoothly on $x \in M$. They are analytic functions of their remaining arguments.

We want to show that for all (u,a) in a certain open set, $A(u,a)$ is a bounded map from $\mathbb{X}_1$ to $\mathbb{X}_0$. Since $\mathbb{X}_1$ is a dense subspace of $\mathbb{E}_1$, we have an extension operator $\tilde{A}$. We will show also that $\tilde{A} : \mathbb{E}_1 \to \mathbb{E}_0$ is a bounded operator. For a fixed $r > 0$, we define the open subsets $\mathfrak{U}_\beta \subset \mathbb{X}_\beta$ and $\mathfrak{U}_\alpha \subset \mathbb{X}_\alpha$ to be

$$\mathfrak{U}_\beta := \{(u,a) : ||u||_{\mathbb{X}_\beta} > r, ||a||_{\mathcal{X}_\beta} > r\},$$

$$\mathcal{U}_\alpha := \mathcal{U}_\beta \cap \mathbb{X}_\alpha.$$

Lemma 5.4.3. *For* $(u, a) \in \mathcal{U}_\beta$, $A(u, a) \in L(\mathbb{X}_1, \mathbb{X}_0)$. *Also for* $(u, a) \in \mathcal{U}_\alpha$, $\tilde{A}(u, a) \in L(\mathbb{E}_1, \mathbb{E}_0)$.

Proof. We follow almost exactly the proof of Lemma 3.3 in [9]. Fix $(u, a) \in \mathcal{U}_\beta$ and let $(v, b) \in \mathbb{X}_1$. Using equation (5.16), consider first the term $a(x, g)^{ij}\partial_i\partial_j v$. We would like to estimate the Hölder norm of this term, so we perform the following manipulation:

$$\begin{aligned} &a(x, g)(\partial^2 v)(x) - a(y, g)(\partial^2 v)(y) \\ &= a(x, g)(\partial^2 v)(x) - a(x, g)(\partial^2 v)(y) + a(x, g)(\partial^2 v)(y) - a(y, g)(\partial^2 v)(y) \\ &= a(x, g)((\partial^2 v)(x) - (\partial^2 v)(y)) + (\partial^2 v)(y)(a(x, g) - a(y, g)). \end{aligned}$$

One can then estimate

$$\begin{aligned} |a(x, g) - a(y, g)| &\leq \int_\gamma |D(a)| ds \\ &\leq \sup |D(a)| dist_h(x, y) \\ &\leq \sup |D(a)| (dist_h(x, y))^\epsilon (1 + (diam_h M)^{1-\epsilon}), \end{aligned}$$

where we integrated along an h-minimizing geodesic. The bounds on u provide corresponding Hölder bounds on g, so we can obtain that a is a polynomial in g of degree N, so that there exists C such that

$$\sup |D(a)| \leq C(1 + ||g||_\epsilon^{N-1})||g||_{1+\epsilon}.$$

Clearly $\frac{|\partial^2 v(x) - \partial^2 v(y)|}{dist_h(x,y)^\epsilon} \leq ||v||_{2+\epsilon}$. Then, using the fact that $\mathfrak{h}^{2\beta+\epsilon} \hookrightarrow \mathfrak{h}^{1+\epsilon} \hookrightarrow \mathfrak{h}^\epsilon$, we have

$$\left|\left|a(x, g)^{ij}\partial_i\partial_j v\right|\right|_\epsilon \leq C_1(1 + ||g||_{\mathbb{X}_\alpha}^N)||v||_{2+\epsilon},$$

for some constant C_1. Similarly, one can obtain estimates on the remaining terms in equation (5.16) in order to see that $A(u, a)$ is indeed a bounded map from $\mathbb{X}_1$ to $\mathbb{X}_0$. The case of $\tilde{A}$ follows in the same manner. □

Let X be a Banach space. A semigroup $S(t) \subset L(X)$ is said to be analytic if $t \mapsto S(t)$ is an analytic map for all $t \in (0, \infty)$. Additionally, $S(t)$ is strongly continuous if $t \mapsto T(t)x$ is continuous on $[0, \infty)$ for all $x \in X$. In order to satisfy assumption (a) and to show that $\tilde{A}$ generates a strongly continuous analytic semigroup on $\mathcal{L}(\mathbb{E}_0)$, we need the following lemma from [19].

Lemma 5.4.4. *$A : D(A) \subset X \to X$ generates a strongly continuous analytic semigroup if A is sectorial and the domain $D(A)$ is dense in X.*

Proof. (Sketch of Proof:) One can show that if A is sectorial (see Definition B.0.7), then the semigroup generated by A, namely $\{e^{tA}\}_{t \geq 0}$, is analytic. Additionally, it can be shown that this semigroup is strongly continuous if and only if

$$\lim_{t \to 0} e^{tA}x = x, \ \forall x \in X.$$

In fact, the statement that $\lim_{t \to 0} e^{tA}x = x$ is equivalent to $x \in \overline{D(A)}$. In other words, the semigroup generated by A is strongly continuous if and only if the domain $D(A)$ is dense in X. (See [19], Proposition 2.1.4 for the detailed proof.) □

We see that we need to show that for $(u,a) \in \mathfrak{U}_\alpha$, $\tilde{A}(u,a)$ is a sectorial operator in $\mathbb{E}_0$. We would like to use the following Proposition from [19].

Proposition 5.4.5. *Let $A : D(A) \subset X \mapsto X$ be a linear operator, and let $\alpha \in (0,1)$. Define $A_\alpha : D_A(\alpha+1) \mapsto D_A(\alpha)$ by $A_\alpha x := Ax$. In other words, A_α is the piece of A defined on $D_A(\alpha)$. Then A_α is a sectorial operator in $D_A(\alpha)$.*

Lemma 5.4.6. *For $(u,a) \in (u,a) \in \mathfrak{U}_\alpha$, $\tilde{A}(u,a) : \mathbb{E}_1 \to \mathbb{E}_0$ is a sectorial operator on $\mathbb{E}_0$.*

Proof. In our case, there exists $\theta \in (0,1)$ such that $\mathbb{X}_0 = D_{\tilde{A}}(\theta)$ and $\mathbb{X}_1 = D_{\tilde{A}}(1+\theta)$. Thus, by Proposition 5.4.5, $A(u,a) : \mathbb{X}_1 \mapsto \mathbb{X}_0$ is a sectorial operator on $\mathbb{X}_0$.

Now let $\eta \in \mathbb{E}_1$, and since $\mathbb{X}_1$ is densely embedded in $\mathbb{X}_1$, let $\{\eta_i\} \in \mathbb{X}_1$ such that $\eta_i \to \eta$ in $\mathbb{E}_1$. Using the notation in Definition B.0.7, fix $\lambda \in S_{\gamma,\omega,A}$. Then we have

$$\begin{aligned} ||R(\lambda,A)\eta_i||_{\mathbb{E}_0} &\leq c||R(\lambda,A)\eta_i||_{\mathbb{X}_0} \\ &\leq \frac{M}{|\gamma-\omega|}||\eta_i||_{\mathbb{X}_1}. \end{aligned}$$

Since the resolvent operator is continuous, and $A = \tilde{A}$ on the dense set $\mathbb{X}_1$, we can pass to the limit to obtain that $\tilde{A}(u,a)$ is sectorial in $\mathbb{E}_0$. □

Lemma 5.4.7. *For $(u,a) \in \mathfrak{U}_\alpha$, $\tilde{A}(u,a) : \mathbb{E}_1 \to \mathbb{E}_0$ generates an analytic semigroup on $L(\mathbb{E}_0)$.*

Proof. This follows from Lemmas 5.4.4 and 5.4.6 □

Remark 5.4.1. For the characterization of the little Hölder spaces as $D_A(\theta)$, see Chapter 3.1.3 of [19].

5.4.1 $\lambda = 0$

We would like to compute the spectrum of L at an Einstein Yang-Mills metric. Since L is self-adjoint, we know that the spectrum is pure point and that it is contained in $\mathbb{R}$. We first consider the case of a flat bundle over a constant curvature Riemann surface ($\lambda = 0$ in the equations above). We consider the linearization of equation (5.5) to be the following operator:

$$L\begin{pmatrix} v \\ b \end{pmatrix} = \begin{pmatrix} \Delta + R_h & 0 \\ 0 & \Delta_d \end{pmatrix}\begin{pmatrix} v \\ b \end{pmatrix}. \tag{5.17}$$

First, we would like to compute the spectrum of L.

Lemma 5.4.8. *If $R_h \leq 0$, then the L^2 spectrum of L is contained in $(-\infty, 0]$. In particular, L is linearly stable.*

Proof. We use the natural L^2 inner product for product spaces

$$(\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} w \\ \rho \end{pmatrix})_{L^2} = \int (v, w) + (b, \rho) d\mu.$$

Then, in the case of $R_h \leq 0$, we have

$$\begin{aligned} (L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix})_{L^2} &= \int ((\Delta v, v) + R_h(v, v) + (\Delta_d b, b)) d\mu \\ &= -||\nabla v||^2 + R_h||v||^2 - ||db||^2 \\ &\leq 0. \end{aligned}$$

We can notice that in the case of $R_h < 0$, the zero eigenvalue of L corresponds to $v = 0$ and b harmonic. Thus our center manifold is $2g$-dimensional, where g is the genus of M. For $R_h = 0$, the zero eigenvalue corresponds to v and b harmonic. So this center manifold will have dimension $1 + 2 = 3$. □

Lemma 5.4.9. *If $R_h > 0$, then $\sigma(L) \cap (0, \infty) \neq \emptyset$. In particular, L is not linearly stable.*

Proof. Since there are no harmonic 1-forms over the 2-sphere, it is clear that there are no unstable directions corresponding to pairs of the form $(0, b)$. Consider pairs of the form $(v, 0)$. Suppose there exists an eigenvalue γ for some $(v_0, 0)$. In this case, we would have

$$\Delta v_0 = (\gamma - 2)v_0 = \mu v_0.$$

The eigenvalues of the laplacian over the n-sphere (having radius 1) are given by $\mu_k = -k(k+n-1)$. In our case, we have $\mu_0 = 0, \mu_1 = -2, \ldots$, so the above equation is equivalent to

$$\gamma - 2 = \mu_0,$$

i.e. $\gamma = 2$. Thus we have a positive eigenvalue for L corresponding to the first eigenvalue of the laplacian. This unstable direction is given by the 1-dimensional space of constant functions. Additionally, we see that $\gamma = 0$ can be obtained by the second eigenvalue. This corresponds to the 2-dimensional space of homogeneous hermitian polynomials of degree 1. [4] □

5.4.2 $\lambda \neq 0$

Now we consider the case where our bundle is not flat. In this case, our operator has the form

$$L \begin{pmatrix} v \\ b \end{pmatrix} = \begin{pmatrix} \Delta v + (R_h - \lambda^2)v & \lambda\langle db, dV\rangle \\ -\lambda\nabla^i v dV_{ij} & \Delta_d b \end{pmatrix}. \tag{5.18}$$

Lemma 5.4.10. $L : L^2(C^\infty(M)) \oplus L^2(\Omega^1(M)) \to L^2(C^\infty(M)) \oplus L^2(\Omega^1(M))$ *is a self-adjoint operator.*

Proof. By definition of the L^2 inner product, we have

$$\begin{aligned}(L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} w \\ c \end{pmatrix})_{L^2} = &\int((\Delta v, w) + (-\lambda^2 + R_h)(v,w) + (\Delta_d b, c))d\mu \\ &+ \int \lambda(db, dV)w + (-\lambda\nabla^i v dV_{ij}, c_j)d\mu.\end{aligned}$$

Clearly the first integral is equal to $\int((v,\Delta w) + (R_h - \lambda^2)(v,w) + (b, \Delta_d c))d\mu$. The second integral becomes

$$\begin{aligned}\int(\lambda(db, dV)w - \lambda\nabla^i v dV_{ij}, c_j)d\mu &= \int(\lambda(db, wdV) + \lambda(d^*(vdV), c))d\mu \\ &= \int(\lambda(b, d^*(wdV) + \lambda(vdV, dc))d\mu \\ &= \int(-\lambda(b_j, \nabla^i w dV_{ij}) + \lambda(dc, dV)v)d\mu.\end{aligned}$$

Thus we see that indeed $(L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} w \\ c \end{pmatrix})_{L^2} = (\begin{pmatrix} v \\ b \end{pmatrix}, L\begin{pmatrix} w \\ c \end{pmatrix})_{L^2}$; i.e. L is in fact self-adjoint. □

Lemma 5.4.11. *If* $R_h \leq 0$*, then* $\sigma(L) \subset (-\infty, 0]$.

Proof. We have the following simple computation:

$$\begin{aligned}
(L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix}) &= (\Delta v, v) + (R_h - \lambda^2)||v||^2 + (\Delta_d b, b) + 2\lambda(db, vdV) \\
&= -||\nabla v||^2 + (R_h - \lambda^2)||v||^2 - ||db||^2 - ||d^*b||^2 \\
&\quad +||db||^2 + \lambda^2||v||^2 \\
&\leq 0,
\end{aligned}$$

where the second line follows from Cauchy-Schwartz.

We claim that $\sigma(L)$ contains zero eigenvalues corresponding to ordered pairs of the form $(0, b)$, where b is harmonic. In particular, b being harmonic implies that $db = 0$, so the computation above implies that such pairs yield a zero eigenvalue. In fact, these are the only zero eigenvalues. In general, we have the following estimate:

$$\begin{aligned}
(L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix}) &\leq -||\nabla v||^2 + (R_h - \lambda^2)||v||^2 - ||db||^2 - ||d^*b||^2 \\
&\quad +||db||^2 + \lambda^2||v||^2 \\
&= -||\nabla v||^2 + R_h||v||^2 - ||d^*b||^2.
\end{aligned}$$

As long as v is not a constant or $d^*b \neq 0$, then $(L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix}) < 0$. If both $v = C$ and $d^*b = 0$ (and $db \neq 0$), then

$$\begin{aligned}
L\begin{pmatrix} v \\ b \end{pmatrix} &= \begin{pmatrix} (R_h - \lambda^2)v & 0 \\ 0 & db \end{pmatrix} \\
&\neq 0\begin{pmatrix} v \\ b \end{pmatrix}.
\end{aligned}$$

If $v = C$ and b is harmonic, then clearly there is no zero eigenvalue. □

Lemma 5.4.12. *If $R_h > 0$, the spectrum of L can be computed in several cases. If $\lambda = \frac{1}{2}$ or 1, then $\sigma(L) \cap (0, \infty) \neq \emptyset$. If $\lambda > 1$, then $\sigma(L) \subset (-\infty, -\delta]$ for some $\delta > 0$.*

Proof. By Lemma 5.3.2, we see that λ can only attain specific values determined by the Chern number of the bundle. The $U(1)$-bundles over S^2 are in 1-1 correspondence with elements of $\mathbb{Z}$ and are determined up to equivalence by their 1^{st} Chern class. In particular, two bundles are equivalent if they have $n \in \mathbb{Z}$ as their Chern number (see, for example, Chapter 6.1 in [20]). Then $\lambda = \frac{n}{2}$, again by Lemma 5.3.2. We first consider the case of $\lambda > 1$. As above, we can compute

$$\begin{aligned}
(L\begin{pmatrix} v \\ b \end{pmatrix}, \begin{pmatrix} v \\ b \end{pmatrix}) &= (\Delta v, v) + (2 - \lambda^2)||v||^2 + (\Delta_d b, b) + 2\lambda(db, vdV) \\
&= -||\nabla v||^2 + (2 - \lambda^2)||v||^2 + \alpha 2\lambda \int (db, vdV) \\
&+ (1-\alpha)2\lambda \int (db, vdV) + \alpha \int (\Delta_d b, b) + (1-\alpha) \int (\Delta_d b, b) \\
&\leq -||\nabla v||^2 + (2 - \lambda^2)||v||^2 + \alpha(\frac{1}{\alpha}||\nabla v||^2 + \frac{\alpha\lambda^2}{4}||b||^2) \\
&+ (1-\alpha)(||db||^2 + \lambda^2||v||^2) - 2\alpha||b||^2 \\
&+ (1-\alpha)(-||db||^2 - ||d^\star b||^2) \\
&\leq (2 - \alpha\lambda^2)||v||^2 + \alpha(\frac{\alpha\lambda^2}{4} - 2)||b||^2,
\end{aligned}$$

where $\alpha \in (0, 1)$ is to be chosen later. We also used Cauchy-Schwartz and the fact that the first eigenvalue of Δ_d acting on 1-forms on S^2 is -2. Then we need to find $\delta > 0$ and $\alpha \in (0, 1)$ such that $2 - \alpha\lambda^2 \leq -\delta$ and $\alpha(\frac{\alpha\lambda^2}{4} - 2) \leq -\delta$.

This amounts to the bounds $\frac{2+\delta}{\alpha} \leq \lambda^2 \leq \frac{8}{\alpha} - \frac{4\delta}{\alpha^2}$. It is clear that for $\lambda^2 > 2$, we can choose such an α and a δ small enough to make these bounds hold. Since λ is quantized, the smallest such value we have to apply these bounds to is $\lambda^2 = \frac{9}{4}$. In these cases, the spectrum is strictly negative.

Let us now consider the remaining two cases: $\lambda = \frac{1}{2}$ and $\lambda = 1$. In both of these cases, we can explicitly show the existence of a positive eigenvalue, corresponding to ordered pairs of the form $(v, 0)$, where v is constant. The computation is the same as that in the $\lambda = 0$ case; we obtain $\mu = \frac{7}{4}$ for $\lambda = \frac{1}{2}$ and $\mu = 1$ for $\lambda = 1$ as eigenvalues for L_λ. □

Remark 5.4.2. In the case of $\lambda = \frac{1}{2}$, we can also show the existence of a positive eigenvalue by treating the off-diagonal pieces of L as a perturbation of L_0. We can then use the perturbation theory of [15]. First, we say that an operator B is A-bounded if $D(A) \subset D(B)$ and if there exist $a, b \in \mathbb{R}$ such that

$$||Bu|| \leq a||u|| + b||Au||, \tag{5.19}$$

for $u \in D(A)$. Notice that if B is a bounded operator, then B is A bounded with $b = 0$. We claim that for $L_{\frac{1}{2}}$, we can write $L_{\frac{1}{2}} = A + B$, where B is A-bounded with $a < 1$ and $b = 0$. Write

$$A \begin{pmatrix} v \\ b \end{pmatrix} = \begin{pmatrix} \Delta v + \frac{7}{4} v & 0 \\ 0 & \Delta b \end{pmatrix},$$

and

$$B \begin{pmatrix} v \\ b \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}(db, dV) \\ -\frac{1}{2}\nabla^i v dV_{ik} & 0 \end{pmatrix}.$$

As a map from $\mathfrak{h}^{1+\epsilon}$, B is bounded. The domain of A is $\mathfrak{h}^{2+\epsilon} \subset D(B)$. Then by considering norms on these spaces, it is clear that inequality (5.19) holds for $a < 1$. We can then apply the following proposition (a combination of Theorem IV-3.18 and comments in Section V-3 of [15]):

Proposition 5.4.13. *Let A be self-adjoint and let B be symmetric and A-bounded with A-bound smaller than 1. Suppose μ is an isolated eigenvalue of A with multiplicity $m < \infty$ and with distance d between μ and $\sigma(A) \setminus \mu$ (this is called the isolation distance). If*

$$a + b(|\mu| + d) < \frac{d}{2},$$

then $A + B$ has exactly m repeated eigenvalues (and no other points of the spectrum) in $(\mu - \frac{d}{2}, \mu + \frac{d}{2})$.

For us, A has an eigenvalue 2 with isolation distance $d = 2$. Thus $A + B$ has an eigenvalue in the interval $(1, 3)$.

In the case of $R_h \leq 0$, we have zero eigenvalues, so we again have the existence of a center manifold and can apply Theorem 5.2.1. For $R_h > 0$ and $\lambda = \frac{1}{2}, 1$, we again have unstable directions corresponding to the constant functions on S^2. For $R_h > 0$ and $\lambda > 1$, we will be able to apply Theorem 5.2.2.

5.5 Stability of the Ricci Yang-Mills Flow

We are finally ready to state the center manifold theorem for the Ricci Yang-Mills flow over surfaces with $R_h \leq 0$.

Theorem 5.5.1. *Let (u, a) be a Yang-Mills connection over a surface with constant curvature $R \leq 0$. The following statements hold:*

1. *$\mathbb{X}_\alpha$ admits the decomposition*

$$\mathbb{X}_\alpha^s \oplus \mathbb{X}_\alpha^c,$$

where $\mathbb{X}_\alpha^c = S(0)$; i.e. $\mathbb{X}_\alpha^c$ corresponds to the algebraic eigenspace of 0.

2. *There exists $d_0 > 0$ such that for all $d \in (0, d_0]$, there is a C^∞ manifold $\mathcal{M}_{loc}^c$ that is locally invariant for solutions of (5.5) as long as they remain in $B(\mathbb{X}_1^c, (u, a), d)$. It is such that $T_{(u,a)}\mathcal{M}_{loc}^c \cong \mathbb{X}^c$. $\mathcal{M}_{loc}^c$ is a unique local center manifold consisting of Einstein Yang-Mills metrics. $\mathcal{M}_{loc}^c$ is 2g-dimensional for $R_h < 0$ and 3-dimensional for $R_h = 0$.*

3. *There are constants $C > 0$, $\omega > 0$, and $d \in (0, d_0]$, such that*

$$||\pi^s((v, b)) - \phi(\pi^c((v, b))||_{\mathbb{X}_1} \leq Ce^{-\omega t}||\pi^s((v(0), b(0)) - \phi(\pi^c((v(0), b(0)))||_{\mathbb{X}_\alpha}$$

for all solutions (v, b) with $(v(0), b(0)) \in B(\mathbb{X}_\alpha, (u, a), d)$ and all times $t \geq 0$ such that the solutions remain in this ball. Here π^s and π^c denote the projections onto $\mathbb{X}_\alpha^s$ and $\mathbb{X}_\alpha^c$ respectively.

Remark 5.5.1. In particular, this theorem states that any bundle that solves (NGRYM) with $(v(0), b(0))$ close enough to an Einstein Yang-Mills manifold will have its conformal factor and connection 1-form converge exponentially fast to those of the EYM manifold.

Proof. We have checked the hypotheses of Theorem 5.2.1, so we obtain the existence of local C^r center manifolds to which solutions to NGRYM that are sufficiently close to (u, a) converge exponentially fast, as long as solutions remain in the given neighborhood of the fixed point. Notice that the family of center manifolds $\mathcal{M}^c_{loc}(r)$ guaranteed by Theorem 5.2.1 are in fact independent of r and consist precisely of EYM connections. If not, then a EYM connection (v, b) sufficiently close to (u, a) would converge exponentially fast to $\mathcal{M}^c_{loc}(r)$. But this contradicts the fact that (v, b) is a fixed point. Since the space of Yang-Mills bundles over a Riemann surface is 2g-dimensional, we see that the center manifolds consist precisely of such pairs. In the case of $R_h = 0$, the local center manifolds again consist of EYM metrics, but we allow the conformal factor to be any constant. So the dimension is 3. The analysis follows in the same way.

Finally, we would like to check that solutions to NGRYM that start in a sufficiently small neighborhood of an EYM actually stay there. Notice that $|\frac{\partial}{\partial t} v| = |R_{e^v h} + r + \frac{1}{2}|F|^2 - \frac{1}{2} f| \leq C_1 e^{-\omega_1 t}$ for some $C_1, \omega_1 > 0$ as long as (v, b) stays in $B(\mathbb{X}_\alpha, (u, a), d)$. Also, $|\frac{\partial}{\partial t} b| = | - d^\star F - dd^\star b| \leq C_2 e^{-\omega_2 t}$ for some $C_2, \omega_2 > 0$ while (v, b) stays in the ball. Let $0 < d' < d$ small such that for all $(\bar{v}, \bar{b})$ with initial data $(\bar{v}, \bar{b})(0) \in B(\mathbb{X}_\alpha, (u, a), d')$,

$$|\bar{v}(t) - u| \leq |\bar{v}(t) - \bar{v}(0)| + |\bar{v}(0) - u| < d,$$

and similarly for $\bar{b}$. These estimates are independent of time, so we see that (v, b) remains in $B(\mathbb{X}_\alpha, (u, a), d)$. The rest of the theorem follows from Theorem 5.2.1. □

Now we consider the case of $R_h > 0$. In this setting, our stability result depends on the value of λ. For $|\lambda| \geq \frac{3}{2}$, we saw that there exists a $\delta > 0$, depending on λ such that $\sigma(L) \subset (-\infty, -\delta]$. So we obtain the following theorem.

Theorem 5.5.2. *Let (u, a) be an Einstein Yang-Mills metric over a surface of constant curvature $R_h > 0$ with Chern number $|c| \geq 3$ and let $\delta_0 \in [0, \delta)$. Then for all $\alpha \in (0, 1)$, there are constants C_α independent of (u, a) and $\hat{d} \in (0, d_0]$ such that, if $(\bar{u}, \bar{a})(0) \in B(\mathbb{X}_\alpha, \hat{d}, (u, a))$, then*

$$||(\bar{u}, \bar{a})(t) - (u, a)||_{\mathbb{X}_1} \leq \frac{C_\alpha}{t^{1-\alpha}} e^{-\delta_0 t} ||(\bar{u}, \bar{a})(0) - (u, a)||_{\mathbb{X}_\alpha}.$$

as long as $(\bar{u}, \bar{a})(t)$ stays in $B(\mathbb{X}_\alpha, \hat{d}, (u, a))$.

Proof. We begin by noting that (u, a) is a unique fixed point, since we have fixed a gauge. Then the proof of the theorem follows in the same fashion as that of Theorem 5.5.1. □

We would like to use a lemma from [9] to show that the convergence of NGRYM implies that of NRYM.

Lemma 5.5.3. (Lemma 3.5, [9]) *Let $V(t)$ be a vector field on a Riemannian manifold $(M^n, g(t))$, where $0 \leq t < \infty$, and suppose there are constants $0 < c \leq C < \infty$ such that*

$$\sup_{x \in M^N} |V(x, t)|_{g(t)} \leq C e^{-ct}.$$

Then the diffeomorphisms ϕ_t generated by V converge exponentially to a fixed diffeomorphism ϕ_∞ of M.

Proposition 5.5.4. *Let (u_0, a_0) be an Einstein Yang-Mills metric with a_0 written in the Coulomb gauge. Suppose there exists a neighborhood $\mathcal{O}$ of (u_0, a_0) measured in the $||\cdot||_{2\alpha+\epsilon}$ norm such that for every $(\tilde{u}_0, \tilde{a}_0) \in \mathcal{O}$, the unique solution $(\bar{u}, \bar{a})$ to NGRYM converges to an Einstein Yang-Mills metric $(\bar{u}_\infty, \bar{a}_\infty)$. Then the unique solution $(\tilde{u}, \tilde{a})$ to NRYM with initial data $(\tilde{u}_0, \tilde{a}_0)$ converges exponentially fast to an Einstein Yang-Mills metric $(\tilde{u}_\infty, \tilde{a}_\infty)$.*

Proof. Since F is invariant under gauge transformation, it is clear that $\tilde{a}_\infty$ is Yang-Mills. So we need to show that $\tilde{a}$ converges to a limit. We have that $\bar{a} \to \bar{a}_\infty$ exponentially fast, so in particular, $d^\star \bar{a} \to 0$ exponentially fast. Thus our vector field W from Lemma 5.3.1 converges to 0 exponentially fast. Our result follows from the previous lemma. □

Appendices

Appendix A

Gauge Theory

We would like to review the necessary notions from gauge theory; in particular, those of principal bundles, connections, and gauge transformations. We follow the notation in [3].

Definition A.0.1. A *principal fiber bundle* (PFB) consists of a manifold P, a Lie group G, a manifold M, and a projection map $\pi : P \to M$ such that the following hold:

1. G acts freely and differentiably on P to the right.

2. The map $\pi : P \to M$ is onto, and the fiber over $x \in M$, $\pi^{-1}(x)$ is diffeomorphic to G. However, as there is no canonical identification of the fiber with G, there is no natural group structure on the fiber.

3. For each $x \in M$ there is an open set U containing x and a diffeomorphism $T_U : \pi^{-1}(U) \to U \times G$ of the form $T_U(p) = (\pi(p), s_U(p))$, where $s_U : \pi^{-1}(U) \to G$ has the property that $s_U(pg) = s_U(p)g$ for all $g \in G$ and for all $p \in \pi^{-1}(U)$. T_U is a *local trivialization.*

Let $p \in P$ and consider T_pP. There is a natural subspace of this vector space given by $V_p := \{X \in T_pP : \pi_*(X) = 0\}$. A *connection* assigns to p a subspace H_p such that $T_pP = V_p \bigoplus H_p$. We require $R_{g*}(H_p) = H_{pg}$, where R_g is right multiplication by the group element g. V_p is called the *vertical subspace*, and H_p is called the *horizontal subspace*. There is an equivalent notion of connection that will also be useful.

Definition A.0.2. Let $\mathcal{G}$ be the Lie algebra of G. A *connection 1-form* is a $\mathcal{G}$-valued 1-form ω defined on P such that the following hold:

1. Let $A \in \mathcal{G}$ and let A^* be the *fundamental vector field* on P defined by

$$A^*_p = \frac{d}{dt}(p\exp(tA))\bigg|_{t=0}. \tag{A.1}$$

   Then $\omega(A^*_p) = A$.

2. $R^*_g\omega = Ad_{g^{-1}}\omega$ for all $g \in G$.

Notice that the horizontal subspace $H_p = \{X \in T_pP : \omega(X) = 0\}$. We can write any vector $X \in T_pP$ as $X = X^V + X^H$, where X^V is vertical and X^H is horizontal.

Let X be a vector field on M. There is a uniquely defined vector field $\tilde{X}$ on P such that $\omega(\tilde{X}) = 0$ and $\pi_*(\tilde{X}_p) = X_{\pi(p)}$ for all $p \in P$. $\tilde{X}$ is called the *horizontal lift* of X. Notice that $R_{g*}\tilde{X} = \tilde{X}$ for all $g \in G$.

Lemma A.0.5. *Let ω be a connection on P, and let $X_p, Y_p \in H_p$ for some $p \in P$. Then $[X_p, Y_p] \in H_p$.*

Proof. Let $x = \pi(p)$. $\pi_* : H_p \to T_xM$ is an isomorphism. So $[X_p, Y_p] = [\pi_*^{-1}\pi_*X_p, \pi_*^{-1}\pi_*Y_p] = \pi_*^{-1}[\pi_*X_p, \pi_*Y_p] = \pi_*^{-1}[\underline{X}_x, \underline{Y}_x] \in H_p$. □

A connection yields a covariant derivative for $\phi \in \Lambda^k(P, \mathcal{G})$, where $\Lambda^k(P, \mathcal{G})$ is the graded Lie algebra of all k-forms on P with values in $\mathcal{G}$. Notice that $\omega \in \Lambda^1(P, \mathcal{G})$. We will also define a certain subset of the $\mathcal{G}$-valued k-forms that obey certain transformation properties.

Definition A.0.3. Let $\bar{\Lambda}^k(P, \mathcal{G})$ be the space of $\mathcal{G}$-valued differential k-forms ϕ on P such that the following hold:

1. For $X_1, \ldots, X_k \in T_pP$,

$$\phi(R_{g*}X_1, \ldots, R_{g*}X_k) = g^{-1}\phi(X_1, \ldots, X_k).$$

2. If one of $X_1, \ldots, X_k$ is vertical, then $\phi(X_1, \ldots, X_k) = 0$.

Definition A.0.4. The *exterior covariant derivative* of $\phi \in \Lambda^k(P, \mathcal{G})$ is

$$D^\omega := (d\phi)^H \in \Lambda^{k+1}(P, \mathcal{G}),$$

where $\phi^H(X_1, \ldots, X_k) = \phi(X_1^H, \ldots, X_k^H)$.

The *curvature* of the connection is then $\Omega := D^\omega \omega \in \Lambda^2(P, \mathcal{G})$. Notice also that $\Omega \in \bar{\Lambda}^2(P, \mathcal{G})$.

Since H_p is isomorphic to $T_{\pi(p)}M$, the star operator on M induces one on H_p; this can be uniquely extended to define a star operator $\bar{\star} : \bar{\Lambda}^k(P, \mathcal{G}) \to \bar{\Lambda}^{n-k}(P, \mathcal{G})$.

Definition A.0.5. The *covariant codifferential* $\delta^\omega : \bar{\Lambda}^k(P, \mathcal{G}) \to \bar{\Lambda}^{k-1}(P, \mathcal{G})$ is defined by $\delta^\omega \phi = (-1)^{nk} \bar{\star} D^\omega (\bar{\star}\phi)$.

Finally, we would like to define the notion of gauge transformations. We will define two related notions.

Definition A.0.6. 1. An *automorphism* of P is a diffeomorphism $f : P \to P$ such that $f(pg) = f(p)g$ for all $g \in G$, $p \in P$. A *gauge transformation* is an automorphism f such that f covers the identity map on M, i.e. $\pi(p) = \pi(f(p))$. We denote the space of gauge transformations as $GA(P)$.

2. Let $C(P,G)$ be the space of all maps $\tau : P \to G$ such that $\tau(pg) = g^{-1}\tau(p)g$. Notice that $C(P,G)$ is naturally isomorphic to the space of sections of the associated bundle $P \times_G G \to M$.

Lemma A.0.6. *$GA(P)$ is isomorphic to $C(P,G)$.*

Proof. Let $\tau \in C(P,G)$, then we can define $f : P \to P$ to be $f(p) = p\tau(p)$. We can see that

$$\begin{aligned} f(pg) &= pg\tau(pg) \\ &= pgg^{-1}\tau(p)g \\ &= p\tau(p)g \\ &= f(p)g, \end{aligned}$$

so $f \in GA(P)$. On the other hand, if $f \in GA(P)$, then define $\tau : P \to G$ via the relationship $f(p) = p\tau(p)$. Then we have

$$\begin{aligned} pg\tau(pg) &= f(pg) \\ &= f(p)g \\ &= p\tau(p)g, \end{aligned}$$

so $\tau(pg) = g^{-1}\tau(p)g$ and hence $\tau \in C(P, G)$. □

We will need to know how gauge transformations act on vector fields. We will assume that G is a matrix group.

Lemma A.0.7. *Let $f \in GA(P)$. If $A \in \mathcal{G}$ and A^* is the corresponding fundamental vector field, then $f_* A^* = A^*$.*

Proof.

$$\begin{aligned} f_* A^*_p &= \frac{d}{dt} f(p \exp tA)\bigg|_{t=0} \\ &= \frac{d}{dt} f(p) \exp tA\bigg|_{t=0} \\ &= A^*_{f(p)}. \end{aligned}$$

□

Lemma A.0.8. *Let $f \in GA(P)$ and $\tau \in C(P, G)$ be related by $f(p) = p\tau(p)$. Then for $X \in T_pP$, we have*

$$f_*(X) = (\tau(p)^{-1} d\tau(X))^*_{f(p)} + R_{\tau(p)*}(X).$$

Appendix B

Maximal Regularity Theory

We use the theories of maximal regularity and analytic semigroups to study the stability of the Ricci Yang-Mills flow at fixed points. Many people have studied these problems; we refer the reader to [19] for an overview of the theory as well as a very complete bibliography.

Consider the linear Cauchy problem

$$u'(t) = Au(t) + f(t), 0 \leq t \leq T \tag{B.1a}$$

$$u(0) = 0, \tag{B.1b}$$

where $A : D(A) \subset X \to X$ is a linear map on Banach spaces. We would like to use function spaces on which u' and Au have the same regularity as f. Analogous to the variations of constants formula (Chapter 5, Example 3), we would like a solution to equation (B.1) to be of the form

$$u(t) = Sf(t) = \int_0^t e^{(t-s)A} f(s) ds. \tag{B.2}$$

In order to make equation (B.2) precise, one must first define what is meant by e^{tA}.

Definition B.0.7. A is *sectorial* if there exist constants $\omega \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi)$,and $M > 0$ such that

i. $\rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,

ii. $||R(\lambda, A)||_{L(X)} \leq \frac{M}{|\lambda-\omega|}, \forall \lambda \in S_{\theta,\omega}$.

Such an A allows one make sense of e^{tA} via the Dunford integral

$$e^{tA} = \frac{1}{2\pi i}\int_{\gamma} e^{t\lambda} R(\lambda, A) d\lambda, t > 0, \tag{B.3a}$$

$$e^{0A}x = x. \tag{B.3b}$$

It can be shown (see, for example, [19]) that if A is sectorial, then e^{tA}, $t \geq 0$, is an analytic semigroup. Namely, $\lim_{t\to 0} ||e^{tA}x - x|| = 0 \ \forall x \in D(A)$, the map $t \mapsto e^{tA}$ is analytic, and $e^{(t+s)A} = e^{tA}e^{sA}$, $\forall t, s \geq 0$.

Additionally, one would like to know when $Sf(t)$ given by equation (B.2) is actually a solution to equation (B.1). For example, if $f \in C([0,T], D(A))$, then Sf defines a solution $u(t) \in C([0,T], D(A)) \cap C^1([0,T], X)$. However, if one tries to relax this regularity by instead letting $f \in C([0,T], X)$, then $Sf \in C^{\theta}([0,T], X) \cap C([0,T], X_{\theta})$, for some $\theta < 1$. Hence Sf does not define a solution to equation (B.1).

One way to resolve this difficulty is to carefully define interpolation spaces between X and $D(A)$. It can be shown that $||tAe^{tA}x||$ is bounded on $(0,1)$ for all $x \in X$ and that this quantity goes to 0 as $t \to 0$, if $x \in \overline{D(A)}$. On the other hand, if $x \in D(A)$, then $||Ae^{tA}x||$ is bounded on $(0,1)$. Thus one defines the interpolation spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ to be

$$D_A(\alpha, p) = \{x \in X : t \mapsto ||t^{1-\alpha-\frac{1}{p}} Ae^{tA}x|| \in L^p(0,1)\},$$

$$D_A(\alpha) = \{x \in D_A(\alpha, \infty) : lim_{t \to 0} t^{1-\alpha} A e^{tA} x = 0\}.$$

The norm on $D_A(\alpha, p)$ is given by

$$||x||_{D_A(\alpha,p)} = ||x|| + [x]_{D_A(\alpha,p)} = ||x|| + ||t^{1-\alpha-\frac{1}{p}} A e^{tA} x||_{L^p(0,1)}.$$

Using these interpolation spaces, one can show that if $f \in C([0,T], D_A(\alpha))$, then Sf defines a solution $u(t) \in C([0,T], D_A(\alpha+1)) \cap C^1([0,T], D_A(\alpha))$. Here $D_A(\alpha+1)$ means that $Ae^{tA}x$ has values in $D_A(\alpha)$. (See Chapter 4 of [19] for a more detailed treatment of maximal regularity results.)

Remark B.0.2. A result of Baillon shows that spaces with maximal regularity contain a subspace isomorphic to the Banach space c_0 of sequences tending to zero in the sup norm [2]. If the target space of our map A is reflexive, then one cannot use maximal regularity theory. This motivates our introduction of the little Hölder spaces.

Appendix C

Center Manifolds for Quasilinear Parabolic Equations

We would like to provide more details concerning the statement and proof of Theorem 5.2.1. Namely, we would like to give an idea of how Simonett constructed the locally invariant manifolds in [24].

Consider the autonomous quasilinear equation

$$\dot{u} + A(u)u = F(u), \ t > 0, \tag{C.1}$$

where $F(0) = 0$ by assumption. Suppose that we are in the setting of Theorem 5.2.1. In particular, we have the Banach spaces $X_1 \hookrightarrow X_0$ and $E_1 \hookrightarrow E_0$. We also have $0 < \beta < \alpha < 1$ fixed and the interpolation spaces X_β, X_α with open subsets G_β, G_α, respectively. Since we are considering a slightly more general case than that of Theorem 5.2.1, we also assume

$$(A, F) \in C^k(G_\beta, \mathcal{L}(X_1, X_0) \times X_0), \ k \in \mathbb{N}, k \geq 1.$$

(For our purposes, $F \equiv 0$.) One can then show that $A(x) \in M_\alpha(X_1, X_0)$, for all $x \in G_\alpha$. This is a maximal regularity result stating that the Cauchy problem

$$\partial_t u + Au = f(t)$$
$$u(0) = x$$

has a unique solution for all $(f, x) \in C_\alpha((0, T], X_0) \times X_\alpha$. The solution

$$u := (\partial_t + A, R_\alpha)^{-1}(f, x)$$

is an element of $V_\alpha((0, T]; X_0, X_1)$. Here $R_\alpha u = u(0)$, and

$$\begin{aligned} V_\alpha((0, T]; X_0, X_1) :=& \{u \in C^1((0, T], X_0) \cap C((0, T], X_1) : \\ & \lim_{t \to 0} t^{1-\alpha}(||u'(t)||_{X_0} + ||u(t)||_{X_1} = 0\}. \end{aligned}$$

We want to study the behavior of equation (C.1) in a neighborhood of the fixed point 0. As we have alluded to earlier, this analysis will be classified by the spectrum of the linearized flow. We will let the linearized operator L be given by

$$L = A(0) - \partial F(0). \tag{C.2}$$

Assume that we can decompose the spectrum of $-L$ as follows:

$$\sigma(-L) = \sigma_s \cup \sigma_c,$$

where $\sigma_s \subset \{\Re(\lambda) < 0\}$ and $\sigma_c \subset i\Re$. Suppose additionally that σ_c consists of finitely many eigenvalues with finite multiplicity. Notice that in the case of the Ricci Yang-Mills flow, the operator L is self-adjoint, thus its spectrum is pure point. In that case σ_c will consist precisely of the zero eigenvalues.

The decomposition of the spectrum of $-L$ leads to a decomposition of both X_0 and X_1. Namely, let

$$X_i = X_i^c \oplus X_i^s,$$

for $i = 0, 1$. Here $X_i^c = \pi^c(X_i)$ and $X_i^s = \pi^s(X_i)$, where π^c is the projection for σ_c and $\pi^s = id_{X_i} - \pi^c$. L also decomposes into $L = L_c \oplus L_s$, where $L_c = L|_{X^c}$ and similarly for L_s. One can then show that $L_s \in M_\alpha(X_1^s, X_0^s)$ and that $-L_s$ generates an analytic C_0-semigroup on X_0^s.

Let $g(z) = (A(0) - A(z))z + F(z) - \partial F(0)z$ for $z \in X_1$. Notice then that equation (C.1) can be rewritten as

$$\dot{u}(t) + Lu(t) = g(u(t)). \tag{C.3}$$

The maximal regularity results from above yield similar such results for g.

Choose ρ such that $W_1(\rho) := B_\rho^c(0) \times B_\rho^1(0) \subset U^c \times U_1^s$. Here $B_\rho^c(0)$ denotes the ball of radius ρ_0 centered at 0 measured with respect to the norm on X^c. We can define a certain "cutting" function r_ρ that gives a modified function $g_\rho = g \circ r_\rho$ which agrees with g on $V = B_\rho^c(0) \times G_1^s$. As long as solutions to equation (C.3) remain inside V, they coincide with solutions to the system

$$\dot{x}(t) + L_c x(t) = \pi^c g_\rho(x, y) \tag{C.4a}$$

$$\dot{y}(t) + L_s y(t) = \pi^s g_\rho(x, y), \tag{C.4b}$$

with initial data $x(0) = \pi^c u_0$ and $y(0) = \pi^s u_0$. Simonett proves a result involving the existence of both globally and locally invariant manifolds. We are only interested in the locally invariant case. For a fixed k, Theorem 4.1 in [24] implies the existence of a map $\psi : X^c \to X_1^s$ that has bounded and continuous derivatives up to order k. If $z(\cdot) = z(\cdot, x, \psi, \rho)$ is a solution of the reduced ODE

$$\dot{z}(t) + L_c z(t) = \pi^c g_\rho(z(t), \psi(z(t))),$$

then one can use a fixed point argument to show that

$$\psi = \int_{-\infty}^{0} e^{\tau L_s} \pi^s g_\rho(z(\tau, x), \psi(z(\tau, x))) d\tau.$$

Then the graph of the function ψ restricted to the ball $B_\rho^c(0)$, which we will call M_{loc}^c, is such that solutions to equation (C.3) that have initial data in M_{loc}^c stay in M_{loc}^c as long as they remain in $W_1(\rho)$. In other words, M_{loc}^c is a locally invariant C^k-manifold. Notice also that the dimension of M_{loc}^c is that of X^c (and hence finite) and that one can parametrize it by $x \mapsto (x, \psi(x))$. Thus the tangent space to M_{loc}^c at 0 is $T_0(M_{loc}^c) = im(id_{X^c}, \partial\psi(0)) = X^c \times \{0\} = X^c$. Hence M_{loc}^c is a local center manifold for equation (C.1).

According to Remark 5.9b, one can show that if X is a Banach space without the properties of maximal regularity but that is sandwiched between two such spaces, then the results of Simonett still hold.

To be precise, let X be a Banach space such that

$$X_1 \hookrightarrow X \hookrightarrow X_\alpha$$

and that $(t, x) \mapsto u(t, x)$ generates a semiflow on $U \cap X$. Then the existence and attractivity of the center manifolds holds:

$$||\pi^s u(t) - \psi(\pi^c u(t))||_X \leq c \frac{N_\alpha}{t^{1-\alpha}} e^{-\omega t} ||\pi^s u_0 - \psi(\pi^c u_0)||_X$$

for all $u_0 \in U_\alpha \cap X$.

In particular, for the case of Ricci Yang-Mills flow, we could expand our results to hold on the space W_p^1. In order to do so, we would have to work on a hierarchy of Besov spaces and Bessel potential spaces.

Appendix D

Little Hölder Spaces

In order to use the maximal regularity theory, we use spaces that are suitable for this context. These are the little Hölder spaces $\mathfrak{h}^{r+\rho}$. Recall that the C^∞ functions are not dense in the Hölder space $C^{r,\rho}$, so we want to work in a slightly smaller space. We define the little Hölder space $\mathfrak{h}^{k+\alpha}$ to be the closure of the C^∞ functions with respect to the $||\cdot||_{k+\alpha}$ norm. First recall the definition of the Hölder space C^α of functions:

$$C^\alpha = \{f \in C_b(I;X) : [f]_{C^\alpha(I;X)} := \sup_{\substack{t,s\in I \\ |t-s|<\delta}} \frac{||f(t)-f(s)||}{|t-s|^\alpha} < \infty\}.$$

We then define the little Hölder space of functions to be

$$\begin{aligned} \mathfrak{h}^\alpha &= \{f \in C^\alpha(I;X) : \lim_{\delta\to 0} \sup_{\substack{t,s\in I \\ |t-s|<\delta}} \frac{||f(t)-f(s)||}{|t-s|^\alpha} = 0\}, \\ \mathfrak{h}^{k+\alpha} &= \{f \in C_b^k(I;X) : f^{(k)} \in \mathfrak{h}^\alpha(I;X)\}. \end{aligned}$$

Example 5. Let $\alpha \in (0,1)$. One can then show that $x^\alpha \in C^\alpha \setminus \mathfrak{h}^\alpha$. To see why this is so, it is sufficient to show that $|x^\alpha - y^\alpha| \leq |x-y|^\alpha$ for $x, y > 0$. Without loss of generality, suppose that $x \geq y > 0$ and define

$$f(x) := x^\alpha - y^\alpha - (x-y)^\alpha.$$

Then $h(y) = 0$ and $h'(x) = \alpha(x^{\alpha-1} - (x-y)^{\alpha-1}) < 0$. Thus, $h(x) \geq h(y) = 0$, so $x^\alpha - y^\alpha \leq (x-y)^\alpha$. Then $x^\alpha \in C^\alpha$.

Now suppose $\delta \in (0,1)$. Let $x = \delta$ and $y = \frac{\delta}{2} \in (0,1)$. Then

$$\frac{|x^\alpha - y^\alpha|}{|x-y|^\alpha} = \frac{\delta^\alpha - (\frac{\delta}{2})^\alpha}{(\frac{\delta}{2})^\alpha} = 2^\alpha - 1 \neq 0.$$

Thus, $x^\alpha \notin \mathfrak{h}^\alpha$.

We can extend this definition to the space of 1-forms on a compact manifold. Let $\mathcal{M}$ be a compact Riemannian manifold. Fix a background metric $\hat{g}$ and a finite atlas $\{U_\upsilon\}_{1\leq\upsilon\leq\Upsilon}$ of coordinate charts covering $\mathcal{M}$. For each $k \in \mathbb{N}$ and $\alpha \in (0,1]$, let $\mathfrak{h}^{k+\alpha}$ denote the little Hölder space of 1-forms with norm $\|\cdot\|_{k+\alpha}$ derived from

$$\|a\|_{0+\alpha} := \max_{\substack{1\leq i\leq n \\ 1\leq\upsilon\leq\Upsilon}} \sup_{x,y\in U_\upsilon} \frac{|a_i(x) - a_i(y)|}{(d_{\hat{g}}(x,y))^\alpha}.$$

We state a few facts about these spaces.

Lemma D.0.9. *For $j < k$ and $0 < \beta < \alpha < 1$, $\mathfrak{h}^{k+\alpha} \hookrightarrow \mathfrak{h}^{j+\beta}$, and this inclusion is continuous and dense.*

Lemma D.0.10. *For $j \leq k \in \mathbb{N}$, $0 < \beta < \alpha < 1$, and $0 < \theta < 1$, if*

$$\theta(k+\alpha) + (1-\theta)(j+\beta)$$

is not an integer, then there is a Banach space isomorphism

$$(\mathfrak{h}^{j+\beta}, \mathfrak{h}^{k+\alpha})_\theta \cong \mathfrak{h}^{(\theta k+(1-\theta)j)+(\theta\alpha+(1-\theta)\beta)}, \tag{D.1}$$

and there exists $C < \infty$ such that for all $\eta \in \mathfrak{h}^{k+\alpha}$

$$||\eta||_{(\mathfrak{h}^{j+\beta},\mathfrak{h}^{k+\alpha})_\theta} \leq C||\eta||^{1-\theta}_{\mathfrak{h}_{j+\beta}}||\eta||^{\theta}_{\mathfrak{h}_{k+\alpha}}. \tag{D.2}$$

Namely, these spaces form a continuous interpolation method.

Let $D_A(\theta)$ be the continuous interpolation spaces. We can show that for certain choices of $D(A)$ and $\mathbb{X}$ these in fact are the little Hölder spaces. Suppose A is given by

$$\begin{cases} D(A) = \{u \in \cap_{p\geq 1} W^{2,p}_{loc} : u, Au \in C(\mathbb{R}^n)\} \\ A : D(A) \mapsto C(\mathbb{R}^n) \end{cases}$$

Theorem D.0.11. *Let $0 < \theta < 1$. Then*

$$D_A(\theta) = \mathfrak{h}^{2\theta}, \text{ if } \theta \neq \frac{1}{2}.$$

Proof. See the proof of Theorem 3.1.12 in [19]. □

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Vita

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