Math 410 (Prof. Bayly) EXAM 1: Wednesday 13 September 2006

There are 5 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

(1)(10 points) Let $A = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$.

(a)(5 points) Evaluate (or state why you cannot) AA, A^TA^T, AA^T and A^TA.

(b)(5 points) The *trace* of a square matrix is the sum of its diagonal elements. Show that

$$trace(A^TA) = trace(AA^T)$$

for any matrix of this form. Do you expect this to be true for matrices of other dimensions?

(2)(20 points) Consider the linear systems $A\vec{x} = \vec{b}_1$ and $A\vec{x} = \vec{b}_2$, where

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \quad \vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \quad \vec{b}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a)(5 points) Using the graph paper on the back of this question sheet, sketch the aiming vectors (columns of A) and target vectors ($\vec{b}_1, \vec{b}_2$). For which target vector do you expect (graphically) there exists solution to the linear system?

(b)(10 points) Find the general solution to the homogeneous system $A^T\vec{u} = \vec{0}$, identify the free variable and null vector $\vec{m}$. Check that $\vec{m}^T\vec{b} = 0$ for whichever of $\vec{b}_1, \vec{b}_2$ the system $A\vec{x} = \vec{b}$ has a solution.

(c)(5 points) Sketch $\vec{m}$ on the same picture as your sketches for (a). Does it look perpendicular to the column vectors?

(3)(25 points) Consider the linear system $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 2 & 1 & 2 & 3 \\ -4 & 1 & -2 & -5 \\ -2 & 5 & 5 & 2 \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

(a)(10 points) Write down the augmented matrix for this system, and use Gaussian elimination to reduce it to echelon form. Identify the lower and upper triangular factors L, U of the coefficient matrix, and check directly that $LU = original\ matrix$.

(b)(15 points) Under what conditions (if any) on $\vec{b}$ does a solution exist? Find the general solution, identify the particular and complementary parts, the free variable(s), and the null vector(s).

(4)(25 points) Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

(a)(5 points) Sketch the aiming vectors (columns of A) and the target vector $\vec{b}$.

(b)(10 points) Find the general solution, identify the particular and complementary solution, the free variable(s), and the null vector(s).

(c)(5 points) The components of the particular solution give a combination of the column vectors that yield the target vector; sketch this combination.

(d)(5 points) Use the components of the null vector to identify a nonzero combination of column vectors that yields the zero vector. Sketch the corresponding cycle of column vectors.

(5)(20 points) Consider the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 & 2 \\ 3 & 2 & 0 & 2 & 2 \\ 2 & -2 & 1 & 2 & 1 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \quad , \quad \vec{m} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

(a)(10 points) Verify that $\vec{m}$ is a null vector of A^T.

(b)(10 points) Under what conditions on $\vec{b}$ is a solution guaranteed to exist?