MATH 410 (BAYLY) EXAM 3 SOLUTIONS (1)

(1) $\vec{x} = \begin{pmatrix} z+2 \\ 1-z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{x}_p} + z \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\vec{x}_c}$

(a) $\vec{n} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(b) $\text{Length}^2(\vec{x}) = (z+2)^2 + (1-z)^2 + z^2$

$= (z^2+4z+4) + (1-2z+z^2) + z^2$

$= 3z^2 + 2z + 5$

(c) $\frac{d}{dz}\left(\text{Length}^2(\vec{x})\right) = 6z+2 = 0$ when $z = \frac{-1}{3}$

(d) Resulting $\vec{x}_{ML} = \begin{pmatrix} 5/3 \\ 4/3 \\ -1/3 \end{pmatrix}$

Dot product $\vec{n}^T \vec{x}_{ML} = (1 \ -1 \ 1) \begin{pmatrix} 5/3 \\ 4/3 \\ -1/3 \end{pmatrix} = \frac{5}{3} - \frac{4}{3} - \frac{1}{3} = 0$ ✓

$\Rightarrow$ ORTHOGONAL!

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(2) $f(\vec{x}) = x^2 + 4xy + 3y^2 + 2xz + 8yz + 2z^2$

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$$f(\vec{x}) = \begin{pmatrix} x & y & z \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{pmatrix}}_{K} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a)

(b) Row-reduce K. $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 2 \end{pmatrix} \xrightarrow[l_{31}=1]{l_{21}=2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

$l_{32} = -2$ $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ This is U, but we're not done yet.

Note $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$, $U = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$

$U = DL^T$

(c) So $f(\vec{x}) = (\vec{x}^T L) D (L^T \vec{x})$

where $L^T \vec{x} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+z \\ y-2z \\ z \end{pmatrix}$

So $f(\vec{x}) = (x+2y+z, \ y-2z, \ z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x+2y+z \\ y-2z \\ z \end{pmatrix}$

$= 1(x+2y+z)^2 - 1(y-2z)^2 + 5z^2$

(2c) Because ~~pivots~~ diagonal entries in D are BOTH positive AND negative, $f(\vec{x})$ is <u>INDEFINITE</u>.

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(3) Data $(1,1)$, $(2,1)$, $(3,3)$, $(4,4)$, $(5,3)$

(a) If $y = \alpha + \beta x$ then

$$1 = \alpha + \beta \qquad 4 = \alpha + 4\beta$$
$$1 = \alpha + 2\beta \qquad 3 = \alpha + 5\beta$$
$$3 = \alpha + 3\beta$$

I.e.
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 3 \end{pmatrix}$$

NO SOLUTION!
Briefly subtracting (1)
1st 2 eqns $\Rightarrow \beta = 0$

while ~~2nd~~ & 3rd $\Rightarrow \beta = 2$ INCONSISTENT

(b) Use Least squares $(A^TA)\vec{x} = A^T\vec{b}$

Here
$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 54 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 43 \end{pmatrix}$$

3b cont so $\begin{pmatrix} 5 & 10 \\ 10 & 54 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 12 \\ 43 \end{pmatrix}$

determines $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ least squares best approx.

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4 a) $E = \begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \end{array}$

b) $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \vec{m}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \vec{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

c) Score difference vector is $\begin{pmatrix} 3 \\ 1 \\ -5 \\ -2 \\ 2 \\ 3 \end{pmatrix} = \vec{D}$

Note $\vec{m}_1^T \vec{D} = -9 \neq 0 \Rightarrow$ NO SOLUTION

and $\vec{m}_2^T \vec{D} = -7 \neq 0$ also $\Rightarrow$ No solution!

(d) Least squares solution would satisfy

$$E^T E \vec{W}_{LS} = E^T \vec{D}$$

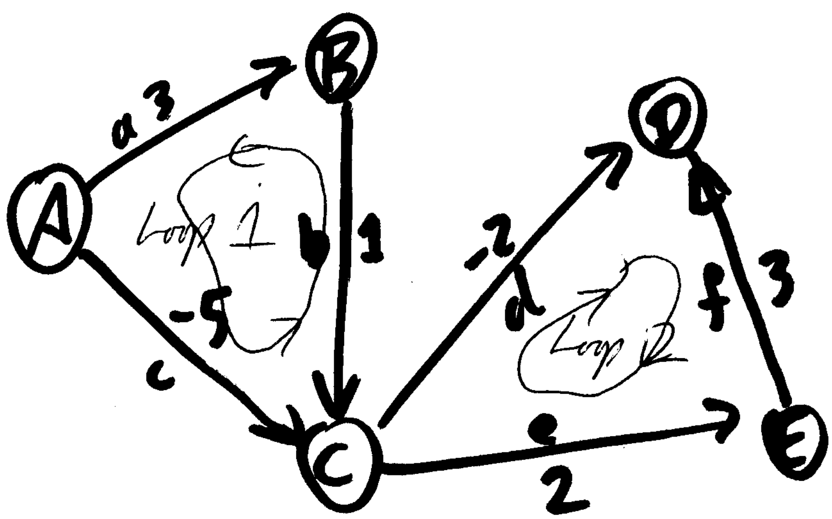
Here

$$E^T E = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$E^T \vec{D} = \begin{pmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -5 \\ -2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ \cancel{6}2 \\ -4 \\ 1 \\ -1 \end{pmatrix}$$

NETWORK FOR PROBLEM 4:

(a)



4e We don't use MIN LENGTH solution even when there's a single free variable, because all the information about ranking can be obtained from the general form. ~~The fact~~ ~~That~~ we when there are 2 free variables it is VERY important to keep them both free, as it allows the possibility of one league being much better than the other. If we used MIN LENGTH we would be making a big mistake in a lot of circumstances!

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5 $A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$ They give us λ's $= 1, -3, 5$

a) $\text{Trace}(A) = 1 - 1 + 3 = 3$ ✓

Sum of evals $= 1 - 3 + 5 = 3$ ✓

b) $\text{Det}(A) = 1 \det\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} - 3 \det\begin{pmatrix} 4 & 4 \\ 2 & 3 \end{pmatrix}$

$= 1(-3) - 3(12 - 8) = -3 - 12 = -15$ ✓

Product of evals $= 1 \times (-3) \times (5) = -15$ ✓

$\lambda=1$ $\quad (A-\lambda I)\vec{x}=\vec{0} \Rightarrow \left(\begin{array}{ccc|c} 0 & 4 & 4 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$

Swap rows 1 & 2 $\quad \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & -2 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

z free $\quad 4y+4z=0 \Rightarrow y=-z$

$3x-2y=0 \Rightarrow x=\frac{2}{3}y=-\frac{2}{3}z$

So $\quad \vec{x}=\begin{pmatrix} -2/3\, z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix} \quad \boxed{\vec{\xi}=\begin{pmatrix} -2/3 \\ -1 \\ 1 \end{pmatrix}}$

(d) $\quad J-\lambda I=\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} \quad \det=\lambda^2+1$

Roots $\lambda=\pm i$

For $\lambda=i$ $\quad (J-\lambda I)\vec{x}=\vec{0} \Rightarrow \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right)$

$l_{21}=i$

$\rightarrow \left(\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad y$ free, $-ix-y=0$

$\Rightarrow -ix=y \quad x=iy$

So $\vec{x}=y\begin{pmatrix} i \\ 1 \end{pmatrix} \quad \boxed{\vec{\xi}=\begin{pmatrix} i \\ 1 \end{pmatrix}}$ is evector belonging to $\lambda=i$

BY CONJUGATION $\boxed{\vec{\xi}=\begin{pmatrix} -i \\ 1 \end{pmatrix}}$ goes with $\lambda=-i$