MATH 410 (BAYLY) HW3,4 SOLUTIONS (Partial)

1.8.1(b) Original system $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & 4 & -2 & -3 \end{array}\right)$

$\xrightarrow{l_{21}=.5} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3.5 & -3.5 & -3.5 \end{array}\right)$ z free $y = z-1$

$2x + (z-1) + 3z = 1$

$\Rightarrow x = 1-2z$ $\vec{x} = \begin{pmatrix} 1-2z \\ z-1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

INFINITE # of solutions! So look for MIN LENGTH:

CALCULUS Way: $\text{Length}^2(\vec{x}) = (1-2z)^2 + (z-1)^2 + z^2$

$L^2(z) = (1-4z+4z^2) + (z^2-2z+1) + z^2$
$= 6z^2 - 6z + 2$

Derivative $L^{2\prime}(z) = 12z - 6 = 0$ if $z = 1/2$

SO $\vec{x}_{ML} = \begin{pmatrix} 0 \\ -1/2 \\ 1/2 \end{pmatrix}$

GEOMETRY WAY: $(AA^T)\vec{u} = \vec{b}$, then $\vec{x}_{ML} = A^T\vec{u}$

Here $AA^T = \begin{pmatrix} 2 & 1 & +3 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 0 & 21 \end{pmatrix}$

Let $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

18.1 (b) cont. SO $\begin{pmatrix}14 & 0\\ 0 & 21\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}1\\ -3\end{pmatrix}$

$\Rightarrow \vec{u}=\begin{pmatrix}1/14\\ -1/7\end{pmatrix}$ $\quad \vec{x}_{ML}=A^T\vec{u} \quad \vec{u}=\begin{pmatrix}1/14\\ -1/7\end{pmatrix}$

$$\vec{x}_{ML}=\begin{pmatrix}2 & 1\\ 1 & 4\\ 3 & -2\end{pmatrix}\begin{pmatrix}1/14\\ -1/7\end{pmatrix}=\begin{pmatrix}\frac{2}{14}-\frac{1}{7}=0\\ 1/14-8/14=-7/14=-1/2\\ 3/14+4/14=7/14=+1/2\end{pmatrix}$$

YES! Same as calculus!

1.8.2 (a) $\begin{pmatrix}6 & 3\\ 4 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}12\\ 9\end{pmatrix}$ No solution!

SO to find LEAST SQUARES Approx solution via NORMAL EQUATIONS $A^TA\vec{x}=A^T\vec{b}$ $=\begin{pmatrix}108\\ 54\end{pmatrix}$

$A^TA=\begin{pmatrix}6 & 4\\ 3 & 2\end{pmatrix}\begin{pmatrix}6 & 3\\ 4 & 2\end{pmatrix}=\begin{pmatrix}52 & 26\\ 26 & 13\end{pmatrix}$ $\quad A^T\vec{b}=\begin{pmatrix}6 & 4\\ 3 & 2\end{pmatrix}\begin{pmatrix}12\\ 9\end{pmatrix}$

SO $\left(\begin{array}{cc|c}52 & 26 & 108\\ 26 & 13 & 54\end{array}\right)\xrightarrow{l_{21}=1/2}\left(\begin{array}{cc|c}52 & 26 & 108\\ 0 & 0 & 0\end{array}\right)$

y free! $x+y/2=\frac{108}{52}=27/13$ $\quad x=\frac{27}{13}-\frac{y}{2}$

so $\vec{x}_{LS}=\begin{pmatrix}27/13-y/2\\ y\end{pmatrix}=\begin{pmatrix}27/13\\ 0\end{pmatrix}+y\begin{pmatrix}-1/2\\ 1\end{pmatrix}$

1.8.20(a) cont. NOTICE that $\vec{x}_{LS}$ is NOT UNIQUE! There's ∞ solution for $\vec{x}_{LS}$.

To find min length $\vec{x}_{LS}$, can either use CALCULUS & minimize $\left(\frac{27}{13}-\frac{y}{2}\right)^2+y^2$ (which I assume you can do)

OR use GEOMETRIC Method. Here, since $\vec{x}_{LS}$ satisfies $B\vec{x}_{LS}=\vec{c}$, where $B=A^TA$ & $\vec{c}=A^T\vec{b}$ we need to solve $(BB^T)\vec{u}=\vec{c}$, then $\vec{x}_{LSML}=B^T\vec{u}$

Explicitly $BB^T=\begin{pmatrix}52&26\\26&13\end{pmatrix}\begin{pmatrix}52&26\\26&13\end{pmatrix}=13^2\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}4&2\\2&1\end{pmatrix}$
(for simplicity) $=13^2\begin{pmatrix}20&10\\10&5\end{pmatrix}=5\times13^2\begin{pmatrix}4&2\\2&1\end{pmatrix}$

SO $5\times13^2\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}108\\54\end{pmatrix}=54\begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}4&2\\2&1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\frac{54}{5\times13^2}\begin{pmatrix}2\\1\end{pmatrix}$ $\left(\begin{array}{cc|c}4&2&108/5\times13^2\\2&1&54/5\times13^2\end{array}\right)$.

$L_{21}=\frac{1}{2}\rightarrow\left(\begin{array}{cc|c}4&2&108/5\times13^2\\0&0&0\end{array}\right)$ v free $4u+2v=\frac{108}{5\times13^2}$

SO $u=\frac{27}{5\times13^2}-\frac{v}{2}$ $\vec{u}=\begin{pmatrix}\frac{27}{5\times13^2}-v/2\\v\end{pmatrix}$

NOTE That the $\vec{u}$ we find ~~from~~ while trying to get to $\vec{x}_{LSML}$ is ALSO not unique! There's ∞ solutions for $\vec{u}$.

BUT when we look for $\vec{x}_{LSML} = B^T \vec{u}$ everything settles out:

$$\vec{x}_{LSML} = \begin{pmatrix} 52 & 26 \\ 26 & 13 \end{pmatrix} \begin{pmatrix} 27/5\times 13^2 - v/2 \\ v \end{pmatrix} = \begin{pmatrix} \frac{52\times 27}{5\times 13^2} - 26v + 26v \\ \frac{26\times 27}{5\times 13^2} - 13v + 13v \end{pmatrix}$$

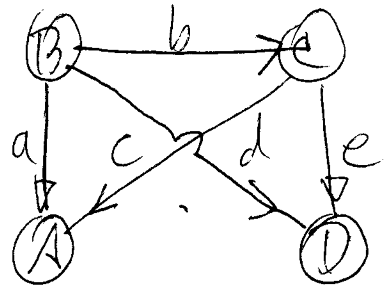
All the v's cancel out!

$\Rightarrow \vec{x}_{LSML}$ really is unique $= \begin{pmatrix} \frac{4\times 27}{5\times 13} = 108/65 \\ \frac{2\times 27}{5\times 13} = 54/65 \end{pmatrix}$

☆ The rest of the Least Squares / Min Length problems will go exactly the same way!

NETWORKS & CIRCUITS

2.6.5



(a)

★ You can work this problem out using row reduction BUT we will use TOPOLOGICAL SHORTCUTS!

EDGE-NODE MATRIX

$$E = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

(b)

<u>Recall:</u> Dim (Nullspace E) = # pieces of network = 1

⇒ row reducing E produces 1 free variable

⇒ ~~RANK~~ 3 columns get pivots ⇒ RANK = 3

(c) Dim (column space E) = RANK = 3

Dim (col space of E^T) = RANK = 3

Dim (nullspace E^T) = # rows − RANK = 2

(d) BASIS of NULL SPACE in E $\vec{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Basis of NULLSPACE E^T is LOOPS (independent):

LOOP $A \to B \to C \to A$ has $\vec{m}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Loop $B \to C \to D \to B$ has $\vec{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

(e) Solvability conditions for $E\vec{x} = \vec{b}$ are $\vec{m}^T \vec{b} = 0$ for $\vec{m}_1, \vec{m}_2$ (KIRCHHOFF VOLTAGE LAWS)

if $\vec{b} = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$ then $\vec{m}_1^T \vec{b} = -a+b+c$ should $=0$

and $\vec{m}_2^T \vec{b} = b-d+e$ should $=0$

(f) A specific vector satisfying these conditions is $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

Now I guess we DO have to row-reduce!

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{l_{31}=1} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow[l_{42}=1]{l_{32}=-1} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\text{and } l_{53}=1]{\text{swap rows } 3,4}$$

$$\to \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

so w free

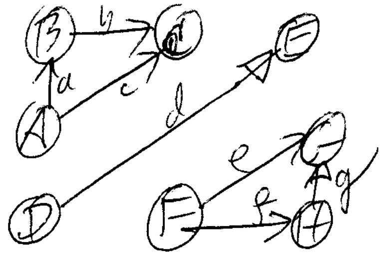
$z = w-1$

$y = w-2$

$x = w$

$$\vec{x} = \begin{pmatrix} w \\ w-2 \\ w-1 \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

2.6.11



has BLOCK - ECHELON MATRIX

$$E = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \left(\begin{array}{ccc|cc|ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right) \end{array}$$

Row reduction on the whole matrix ~~goes~~ just as if each block is reduced by itself.

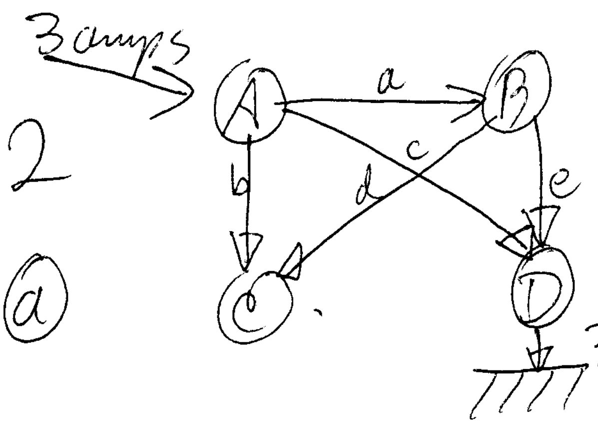
(b) Each block gets $\underline{1}$ free variable, corresponding to columns C, E, and H $\Rightarrow$ dim(Nullspace E) $= 3$

~~Basis~~ Null basis vectors are $\vec{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{n}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\vec{n}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

(c) Each piece of the network gets a null basis vector. Entries are $\underline{1}$ if the node belongs to the piece; 0 otherwise

6.2.2

(a)



$$E = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \end{array}$$

(b) Circuit equations are $E^T C E \vec{V} = \vec{I}_{ext}$

Here all C's $= 1 \Rightarrow C$ matrix $=$ Identity

$$\Rightarrow E^T C E = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} \quad \text{and } \vec{I}_{ext} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -3 \end{pmatrix}$$

(c)

SOLVING
$$\left(\begin{array}{cccc|c} 3 & -1 & -1 & -1 & 3 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & -3 \end{array}\right) \xrightarrow[\substack{l_{41}= \\ -1/3}]{\substack{l_{21}= \\ l_{31}=}} \left(\begin{array}{cccc|c} 3 & -1 & -1 & -1 & 3 \\ 0 & 8/3 & -4/3 & -4/3 & 1 \\ 0 & -4/3 & 5/3 & -1/3 & 1 \\ 0 & -4/3 & -1/3 & 5/3 & -2 \end{array}\right)$$

$$\xrightarrow[l_{42}=-1/2]{l_{32}=-1/2} \left(\begin{array}{cccc|c} 3 & -1 & -1 & -1 & 3 \\ 0 & 8/3 & -\frac{4}{3} & -\frac{4}{3} & 1 \\ 0 & 0 & 1 & -1 & 3/2 \\ 0 & 0 & -1 & 1 & -3/2 \end{array}\right) \xrightarrow{l_{43}=-1} \left(\begin{array}{cccc|c} 3 & -1 & -1 & -1 & 3 \\ 0 & 8/3 & -4/3 & -\frac{4}{3} & 1 \\ 0 & 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $V_D =$ free $\quad V_C = \frac{3}{2} + V_D \quad V_B = \frac{3}{8} + \frac{1}{2}(V_C + V_D)$

$\Rightarrow V_B = \frac{9}{8} + V_D \quad , V_A = 1 + \frac{1}{3}(V_B + V_C + V_D) = \frac{15}{8} + V_D$

(d) Connect bulb across nodes (A) & (D)

6.2.4 Well just set it up until the point of doing the row reduction. ~~These parts~~ BTW you are ALLOWED to use calculators / computers / software packages to do homework calculations!

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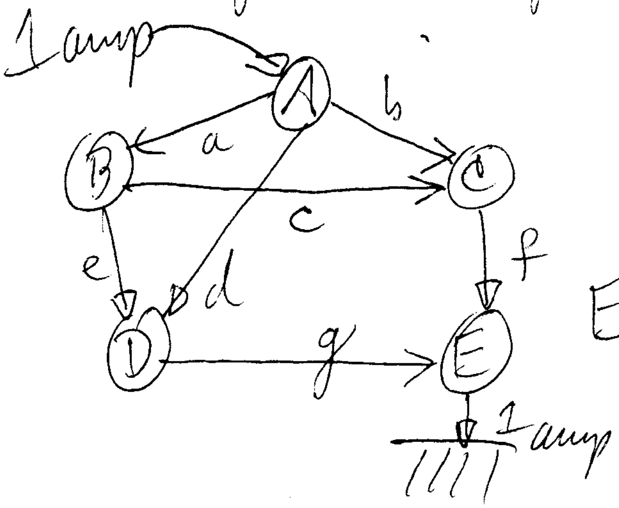


$$E = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

Conductivities are all $= 1$ except $C_b = 1/2$

$$\text{So } E^TCE = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 5/2 & -1 & -1/2 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1/2 & -1 & 5/2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix} \end{array}, \quad \text{also } \vec{I}_{ext} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Note pattern:

Diag elements = SUM of CONDUCTIVITIES of edges attached to that node

OFF-DIAG ELEMENTS = −(conductivity of edge between nodes) if 2 nodes are connected

Or 0 if 2 nodes NOT connected.

Then solve $E^TCE\vec{V} = \vec{I}_{ext}$

Team Ranking

Opening weekend

$$\begin{tikzcd} A \arrow[r, "-3", "a"'] & B \\ C \arrow[r, "+4", "b"'] & D \end{tikzcd}$$

Winning potentials

w_A, w_B, w_C, w_D

$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 1 & 4 \end{array}\right)$$

Echelon form already!

w_B, w_D free

$w_A = w_B + 3 \qquad w_C = w_D - 4$

At this stage we can say that A is better than B by 3 units, D better than C by 4, BUT since w_B and w_D are free we cannot compare w_A & w_B to w_C & w_D.

2nd weekend

$$\begin{tikzcd} A \arrow[r, "-3", "a"'] & B \\ C \arrow[r, "b", "+4"'] & D \arrow[u, "-2", "c"'] \end{tikzcd}$$

Now
$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 1 & 0 & -1 & -2 \end{array}\right)$$

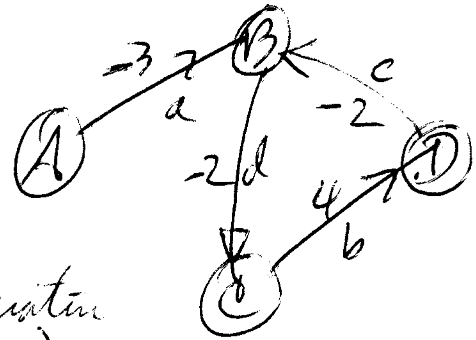
(Echelon form if we swap rows 2 & 3)

~~Also~~
Now only w_D is free &

and $w_C = w_D - 4$, $w_B = w_D - 2$, $w_A = w_D + 1$

SO A is best, then D, then B, then C

3rd weekend:

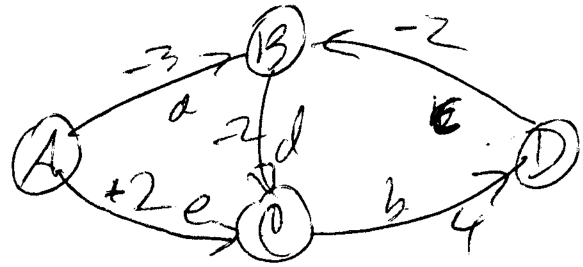


We add the equation

$$\left(\begin{array}{cccc|c} 0 & -1 & 1 & 0 & -2 \end{array}\right) \quad \text{I.e. } W_B = W_C + 2$$

which is CONSISTENT with previous results

4th weekend:



Now A beats B, B beats C, & C beats A

therefore there can be NO solution to $E\vec{W} = \vec{D}$

BUT maybe we can get approx. with least squares approx $E^T E \vec{W} = E^T \vec{D}$

Here

$$E = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \; \& \; \vec{D} = \begin{pmatrix} -3 \\ 4 \\ -2 \\ -2 \\ 2 \end{pmatrix}$$

$$E^T \vec{D} = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 4 \\ -2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -4 \\ 6 \end{pmatrix}$$

$$E^T E = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

So $\left(\begin{array}{cccc|c} 2 & -1 & -1 & 0 & 1 \\ -1 & 3 & -1 & -1 & -3 \\ -1 & -1 & 3 & -1 & -4 \\ 0 & -1 & -1 & 2 & 6 \end{array}\right) \xrightarrow[L_{31}=1/2]{L_{21}=1/2} \left(\begin{array}{cccc|c} 2 & -1 & -1 & 0 & 1 \\ 0 & 5/2 & -3/2 & -1 & -5/2 \\ 0 & -3/2 & 5/2 & -1 & -7/2 \\ 0 & -1 & -1 & 2 & 6 \end{array}\right)$

$\xrightarrow{\text{SWAP}} \left(\begin{array}{cccc|c} 2 & -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 2 & 6 \\ 0 & 0 & -4 & 4 & 29/2 \\ 0 & 0 & 4 & -4 & -29/2 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 2 & -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & 2 & 6 \\ 0 & 0 & -4 & 4 & 29/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

So w_D free again, $w_C = w_D - 29/8$

$w_B = 2w_D - (w_D - 29/8) - 6 = w_D - 23/8$

$w_A = \frac{1}{2} + \frac{1}{2}\left(w_D - \frac{23}{8}\right) + \frac{1}{2}\left(w_D - \frac{29}{8}\right)$

$w_A = w_D + 9/2$

So now w_A leads again, then w_D,

then w_B, then w_C