MATH 410 (BAYLY) SOLUTIONS

Please let me know if I made any arithmetic slips! — B.

(1)

$$2x + y + 3z = -2$$
$$4x + 3y + 7z = -2$$

Augmented matrix $\left(\begin{array}{ccc|c} 2 & 1 & 3 & -2 \\ 4 & 3 & 7 & -2 \end{array}\right)$

$L_{21} = 2$

$$\xrightarrow[\text{FORM}]{\text{ECHELON}} \left(\begin{array}{ccc|c} \boxed{2}^{\text{pivot}} & 1 & 3 & -2 \\ 0 & \boxed{1}_{\text{pivot}} & 1 & 0+2 \end{array}\right) \xrightarrow[\text{JORDAN}]{\text{GAUSS}} \left(\begin{array}{ccc|c} 2 & 0 & 2 & -4 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

z free $\quad y = 2 - z$

$x = -2 - z$

(a) $\vec{x} = \begin{pmatrix} -2-z \\ 2-z \\ z \end{pmatrix}$ general solution

(b) Length^2 of $\vec{x}$ is $(-2-z)^2 + (2-z)^2 + z^2$

$$= (4 + \cancel{4z} + z^2) + (4 - \cancel{4z} + z^2) + z^2$$

$$= 3z^2 + 8$$

$$\frac{d}{dz}(\text{length}^2) = 6z + 0 = 6z$$

Min length when $6z = 0 \Rightarrow z = 0$

$$\vec{x}_{\text{min length}} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

(a)

2 Row reduce $\begin{pmatrix} 1 & -3 \\ -2 & 7 \\ 1 & -1 \end{pmatrix}$ $\quad L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$

$L_{21} = -2$, $L_{31} = 1$

$$\to \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \xrightarrow{L_{32}=2} \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = U$$

CHECK: $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 6+1=7 \\ 1 & -3+2=-1 \end{pmatrix}$ ✓

(b) $L\vec{c} = \vec{b}$ has augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ -2 & 1 & 0 & b_2 \\ 1 & 2 & 1 & b_3 \end{array}\right)$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2+2b_1 \\ 0 & 2 & 1 & b_3-b_1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2+2b_1 \\ 0 & 0 & 1 & (b_3-b_1)-2(b_2+2b_1) \end{array}\right)$$

$$\Rightarrow \boxed{c_1 = b_1, \quad c_2 = b_2+2b_1, \quad c_3 = b_3-2b_2-5b_1}$$

Then $U\vec{x} = \vec{c}$ has augmented matrix $\left(\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 1 & b_2+2b_1 \\ 0 & 0 & b_3-2b_2-5b_1 \end{array}\right)$

NEED $b_3-2b_2-5b_1 = 0$ for SOLUTION!

3) $x + 2ay = 1 \quad ax + 8y = 2$

Augmented matrix $\left(\begin{array}{cc|c} 1 & 2a & 1 \\ a & 8 & 2 \end{array}\right) \xrightarrow{l_{21}=a} \left(\begin{array}{cc|c} 1 & 2a & 1 \\ 0 & 8-2a^2 & 2-a \end{array}\right)$

(a) If $8-2a^2 \neq 0$ then there is always a solution, & it is UNIQUE (no free variables)

If $8-2a^2 = 0 \Rightarrow 2a^2 = 8 \quad a^2 = 4$

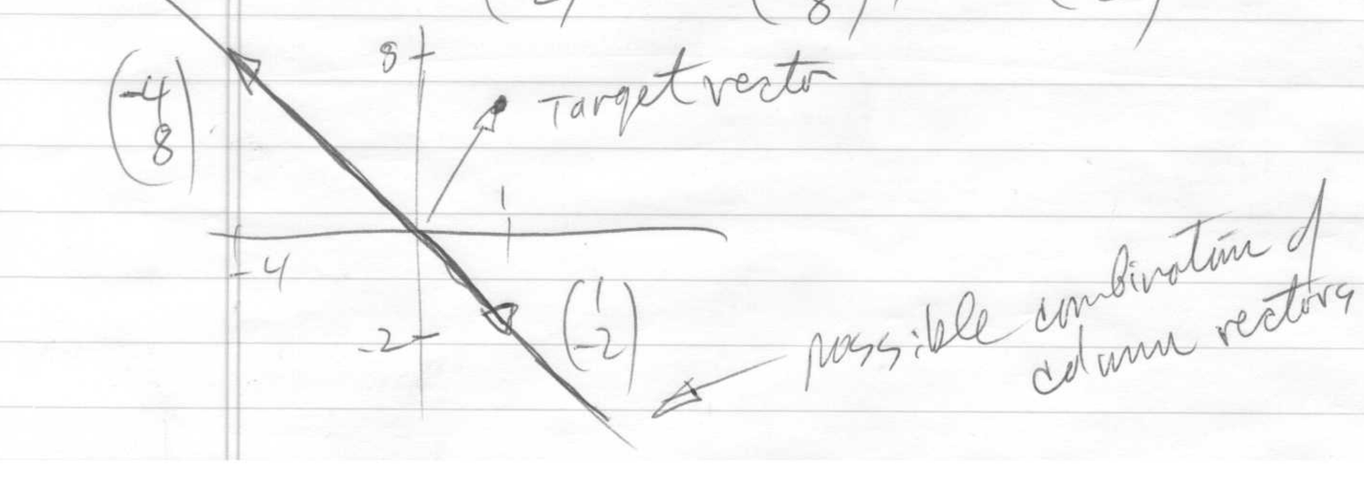
$a = \pm 2$ there could be problems

If $a = 2$, bottom row $\Rightarrow 0 = 0$, and y is free $\Rightarrow \infty$ solutions

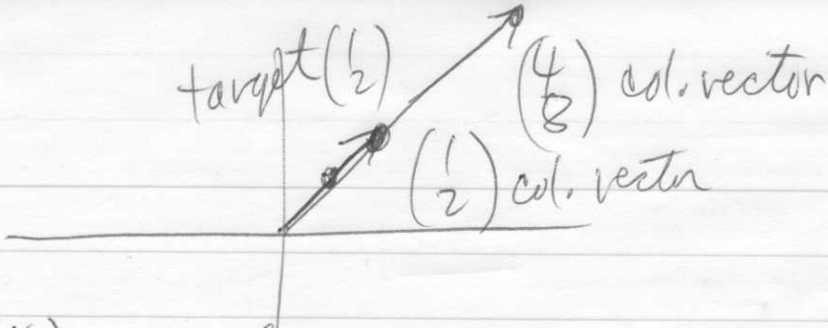
If $a = -2$ bottom row $\Rightarrow 0 = \overset{2-(-2)}{+4}$

$\Rightarrow$ NO SOLUTIONS!

(b) $a = -2$ $\quad \begin{pmatrix} 1 \\ -2 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



3c $a = 2$



$$\begin{pmatrix}1\\2\end{pmatrix} x + \begin{pmatrix}4\\8\end{pmatrix} y = \begin{pmatrix}1\\2\end{pmatrix}$$

Target vector is IN LINE WITH BOTH column vectors!

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4 ⓐ $\|\vec{x}\|_{max} = 2 \qquad \|y\|_{max} = 1$

$$\|A\|_{max} = \max\begin{pmatrix}3+2+1\\2+1+2\end{pmatrix} = 6$$

$$\|B\|_{max} = \max\begin{pmatrix}1+2+2\\4+2+3\end{pmatrix} = 9$$

ⓑ $A\vec{x} = \begin{pmatrix}3 & -2 & -1\\2 & 1 & 2\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}6\\1\end{pmatrix} \qquad \|A\vec{x}\|_{max} = 6$

CHECK $6 \le 6 \times 2 = 12$ ✓

$A\vec{y} = \begin{pmatrix}3 & -2 & -1\\2 & 1 & 2\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}6\\-1\end{pmatrix} \qquad \|Ay\|_{max} = 6 \le 6 \times 1 = 6$

EQUALITY!

4 c In part b, the second vector had the same pattern of + & − signs as the row in the matrix with the biggest sum of absolute values. SO try the same here:

$$\vec{z} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \qquad B\vec{z} = \begin{pmatrix} -1 & -2 & 2 \\ -4 & 2 & -3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

sure enough $\|\vec{z}\|_{max} = 1 \qquad \|B\vec{z}\|_{max} = 9$

$$= \|B\|_{max} \|\vec{z}\|_{max}$$

5 a $A^T = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 2 & 2 & 0 & 2 \end{pmatrix}$

RHS would be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which we can just remember since it does not change.

$L_{21} = 1$

$$\begin{pmatrix} \boxed{2}^{\text{pivot}} & 1 & -2 & 3 \\ 0 & \boxed{1}^{\text{pivot}} & -2 & -1 \end{pmatrix}$$ ECHELON FORM

GAUSS-JORDAN

$$\begin{pmatrix} \cancel{2}^{1} & 0 & -\cancel{2}^{1} & \cancel{4}^{2} \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

Suppose variables are w, x, y, z

Then z free, y free, $x = z - 2y$, $w = y - 2z$

RANK $(A^T) = 2$, Nullity $= 4 - 2 = 2$

In vector form $\vec{x} = \begin{pmatrix} y-2z \\ z-2y \\ y \\ z \end{pmatrix} = y\underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{m}_1} + z\underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{m}_2}$ 6

Here $\vec{m}_1$, $\vec{m}_2$ are NULL VECTORS of A^T

(b) Need $\vec{m}_1^T \vec{b} = 0$, $\vec{m}_2^T \vec{b} = 0$ for $A\vec{x} = \vec{b}$ to have a solution (& similarly for $\vec{c}$, $\vec{d}$)

Here $\vec{m}_1^T \vec{b} = \begin{pmatrix} 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \\ 5 \end{pmatrix} = 2-6+2 = -2 \neq 0$

so $A\vec{x} = \vec{b}$ has NO SOLUTION

$\vec{m}_1^T \vec{c} = \begin{pmatrix} 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ -2 \\ 5 \end{pmatrix} = 4-6-2 = -4 \neq 0$

NO SOLUTION

$\vec{m}_1^T \vec{d} = \begin{pmatrix} 1 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -2 \\ 3 \end{pmatrix} = \cancel{8}\,4 - 10 - 2 = -8 \neq 0$

NO SOLUTION!

☆ I had intended one of these to have a solution. BUT I guess I mis-cooked my numbers!