Math 410 (Prof. Bayly) EXAM 2: Monday 8 March 2004

There are 4 problems on this exam. They are not all the same length, difficulty, or point value. You should read the exam through carefully before deciding which problem to work on first. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

(1)(30 points) Suppose we're given two vectors $\vec{a} = (1, 1, 1, 1)^T$ and $\vec{b} = (-1, 3, -1, 3)^T$.

(a)(10 points) Use the Gram-Schmidt procedure to convert $\vec{a}$ and $\vec{b}$ into a pair of unit length, orthogonal, vectors $\vec{q}_1$ and $\vec{q}_2$ that span the same space.

(b)(5 points) If $A = (\vec{a}, \vec{b})$ is the matrix whose columns are the vectors $\vec{a}$ and $\vec{b}$, and $Q = (\vec{q}_1, \vec{q}_2)$ similarly defined, find an upper triangular matrix R for which $A = QR$.

(c)(5 points) If $\vec{d} = (-2, 6, 2, 2)^T$ and $\vec{x} = (x, y)^T$, convince me there is no solution $\vec{x}$ to $A\vec{x} = \vec{d}$.

(d)(10 points) Calculate $Q^T A$ and $Q^T \vec{d}$, and solve the system $(Q^T A)\vec{x} = Q^T \vec{d}$ for $\vec{x}$. (This will be the least-squares approximate solution to $A\vec{x} = \vec{d}$, but you don't have to show this.)

(2)(30 points) Let $\vec{a} = (2, 1)^T$ and $\vec{b} = (1, 2)^T$. You may leave the answers to this problem in terms of square roots of single-digit integers.

(a)(10 points) Draw a BIG picture (about half a sheet of standard paper) of the x, y plane with these vectors prominently shown, and the parallelogram obtained by joining the tail of $\vec{a}$ to the head of $\vec{b}$, and vice versa. Calculate the lengths of $\vec{a}$, $\vec{b}$, and the cosine of the angle θ between them.

(b)(10 points) Calculate the projection $\vec{p}$ of $\vec{b}$ onto the line through $\vec{a}$, and the orthogonal remainder $\vec{q} = \vec{b} - \vec{p}$. Draw $\vec{p}$ and $\vec{q}$ on the picture from (a), and calculate the length of $\vec{q}$.

(c)(5 points) Recall that the area of a parallelogram is $(base) \times (height)$. Which of the quantities you calculated above is the base, and which the height? What is the area of the parallelogram you drew in (a)?

(d)(5 points) Calculate $det(\vec{a}, \vec{b})$. Does it equal the area you found in (c)?

(3)(20 points) Three teams are in a tournament, and we wish to rank the teams based on a potential or "voltage" measure. In an ideal world the score difference in an actual game would equal the voltage difference between the teams, but in the real world the best we can hope for is a least-squares approximation.

The teams and games can be represented pictorially by a network with directed edges. The vector of voltages $\vec{x}$ ideally satisfies $A\vec{x} = \vec{b}$, where A is the network matrix and $\vec{b}$ the vector of actual score differences.

(a)(8 points) Suppose team (1) beats team (2) by 3 points to 2, and team (2) beats team (3) by 7 points to 6. Draw the corresponding network, formulate $A\vec{x} = \vec{b}$, and find the solution which happens to exist in this case. What's the ranking?

(b)(12 points) Suppose team (3) subsequently beats team (1) by 5 points to 1. Draw the new diagram and formulate the new $A\vec{x} = \vec{b}$, but don't try to solve it because it has no exact solutions. Instead, find the least-squares best approximation and the new ranking.

(4)(20 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, and $\vec{x} = (x, y, z)^T$.

(a)(10 points) Construct the augmented matrix for the system $A\vec{x} = \vec{b}$, and row reduce it to echelon form. Find the general solution, and a basis for the null space and column space of A.

(b)(10 points) A better way (than calculus) to find the minimum-length solution is to seek $\vec{x} = A^T\vec{u}$, where $\vec{u} = (u, v)^T$ satisfies $AA^T\vec{u} = \vec{b}$. Find $\vec{u}$ and the corresponding $\vec{x}$. What value of z in the general solution gives the minimum-length solution?