MATH 410 (BAXLEY) EXAM 2 SOLUTIONS

1

a

$\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\|\vec{a}\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

$\Rightarrow \vec{q}_1 = \dfrac{\vec{a}}{\|\vec{a}\|} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$ [FIRST ~~make~~ get unit vector in direction of $\vec{a}$]

Now we ~~must~~ find the ~~orthogonal~~ projection of $\vec{b}$ orthogonal to $\vec{q}_1$:

$$\vec{q} = \vec{b} - (\vec{q}_1^{\,T}\vec{b})\,\vec{q}_1$$

Here $\vec{q}_1^{\,T}\vec{b} = \left(\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right)\begin{pmatrix} -1 \\ 3 \\ -1 \\ 3 \end{pmatrix} = -\tfrac{1}{2} + \tfrac{3}{2} - \tfrac{1}{2} + \tfrac{3}{2} = 2$

so $\vec{q} = \begin{pmatrix} -1 \\ 3 \\ -1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix}$

Now fix length: $\|\vec{q}\| = \sqrt{2^2+2^2+2^2+2^2} = \sqrt{16} = 4$

$\Rightarrow \vec{q}_2 = \dfrac{\vec{q}}{\|\vec{q}\|} = \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$

b the numbers that go into R are $\|\vec{a}\|$, $\vec{q}_1^{\,T}\vec{b}$, and $\|\vec{q}\|$

$\Rightarrow$ ~~R~~ $R = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$

(1) (c) $A\vec{x} = \vec{d}$ means $\begin{pmatrix} 1 & -1 \\ 1 & 3 \\ 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 2 \\ 2 \end{pmatrix}$

$\left.\begin{array}{l} \text{1st row says } x - y = -2 \\ \text{3rd row says } x - y = +2 \end{array}\right\}$ CANNOT BOTH BE TRUE!

(also 2nd & 4th rows cannot both be true)

(d) $Q^T A = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 3 \\ 1 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix} = R$ as you probably guessed

$Q^T \vec{d} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -2 \\ 6 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

So $\vec{x}$ satisfies $\begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

2nd row $\Rightarrow 4y = 4 \Rightarrow y = 1$

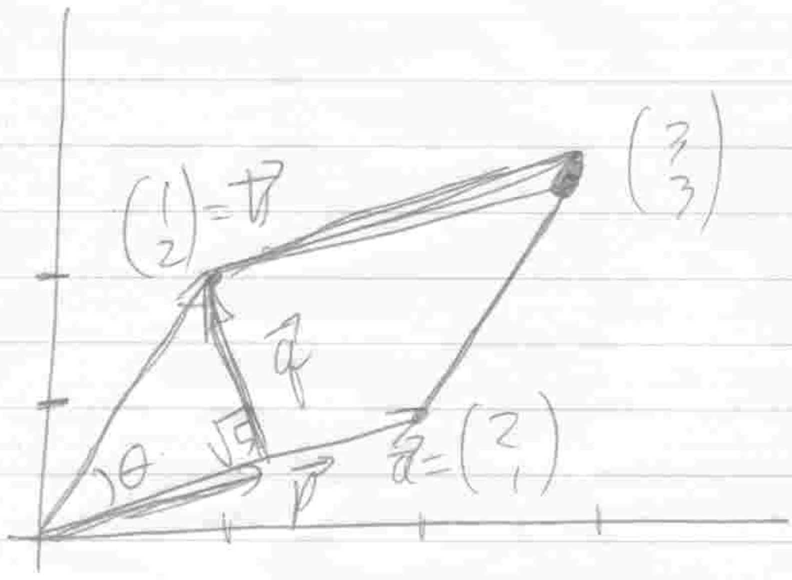
1st row $\Rightarrow 2x + 2y = 4 \Rightarrow 2x = 2$ $x = 1$

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

③

②

ⓐ



X WRONG! THIS PICTURE IS TOO SMALL! NO POINTS!

$$\|\vec{a}\| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\|\vec{b}\| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\cos\theta = \frac{\vec{a}^T\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{1\cdot 2 + 2\cdot 1}{(\sqrt{5})(\sqrt{5})} = \frac{4}{5}$$

ⓑ $$\vec{p} = \vec{a}\,(\vec{a}^T\vec{a})^{-1}(\vec{a}^T\vec{b})$$

$$= \begin{pmatrix}2\\1\end{pmatrix}\frac{1}{5}(4) = \begin{pmatrix}8/5\\4/5\end{pmatrix}$$

$$\vec{q} = \vec{b} - \vec{p} = \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}8/5\\4/5\end{pmatrix} = \begin{pmatrix}-3/5\\6/5\end{pmatrix} = \frac{3}{5}\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\|\vec{q}\| = \sqrt{\frac{9}{25}+\frac{36}{25}} = \sqrt{\frac{45}{25}} = \sqrt{\frac{9}{5}} = \frac{3}{\sqrt{5}}$$

ⓒ (over)

2 c) Base $= \|\vec{a}\| = \sqrt{5}$, height $= \|\vec{q}\| = \frac{3}{\sqrt{5}}$ (4)

$\text{Area} = \sqrt{5} \cdot \frac{3}{\sqrt{5}} = 3$

d) $\det(\vec{a}\ \vec{b}) = \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 2\cdot 2 - 1\cdot 1 = 3$

YES! $\text{Area} = \det(\vec{a}\ \vec{b})$.

3 a) If network is

for example

Graph matrix

$$A = \begin{array}{c|ccc} \text{game} & 1 & 2 & 3 \ \text{team} \\ a & -1 & 1 & 0 \\ b & 0 & 1 & -1 \end{array}$$

$A\vec{x} = \vec{b}$ is $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ +1 \end{pmatrix} \begin{matrix} = 2-3 \\ = 7-6 \end{matrix}$

z is free

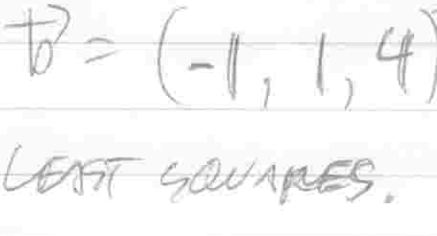
$y - z = 1 \Rightarrow y = 1 + z$

$-x + y = -1 \Rightarrow -x + (1+z) = -1 \quad x = 2 + z$

so $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Team 1 highest
Team 2 middle
Team 3 last.

b) Now network is



matrix is $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$\vec{b} = (-1, 1, 4)^T$

supposedly NO solution, so use LEAST SQUARES.

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3b cont $A^T A = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$A^T \vec{b} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$

AUGMENTED MATRIX

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & -3 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & -1 & -3 \\ 0 & 3/2 & -3/2 & -3/2 \\ 0 & -3/2 & 3/2 & 3/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & -1 & -1 & -3 \\ 0 & 3/2 & -3/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \text{OK!}$$

$\boxed{z \text{ free,}}$ $\frac{3}{2}y - \frac{3}{2}z = -3/2 \Rightarrow \boxed{y = z - 1}$

$2x - (z-1) - (z) = -3$

$\Rightarrow 2x = 2z - 4$ $\boxed{x = z - 2}$

so NOW $\vec{x} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Team 1 LAST!
Team 2 second!
Team 3 FIRST!

4

~~[illegible]~~

(a) Augmented matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 6 \end{array}\right)$

row reduces to $\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & \boxed{-3} & -3 & -6 \end{array}\right)$ in 1 step!

pivots

General solution: z free, $-3y-3z=-6$

$\boxed{y=-z+2=2-z}$

$x+y+z=6 \Rightarrow x+(-z+2)+z=6$

$x+2=6$ $\boxed{x=4}$

$$\boxed{\text{General } \vec{x} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

null vector $\vec{n}$

BASIS for COLUMN SPACE = columns 1, 2 (with pivots)

$= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

BASIS for NULL SPACE is $\vec{n} = (0\ -1\ 1)^T$

(b) $AA^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$

$AA^T\vec{u} = \vec{b}$ becomes $\begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \Rightarrow \begin{matrix} u=2 \\ v=1 \end{matrix}$

$\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Then $\vec{x}_{\text{min length}} = A^T \vec{u} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$

Sure enough, min length $\vec{x}$ = general $\vec{x}$ with $z = 1$

Interesting note (not asked on exam): Recall that

$\vec{n}^T(\vec{x}_{\text{min length}})$ is supposed to $= 0$

Here $\vec{n}^T \vec{x}_{\text{min length}} = (0\ -1\ 1) \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 0$ ✓ yes!