MATH 410 (BAYLY) EXAM 3 SOLUTIONS

1 a $A = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$ $\quad p_{char}(\lambda) = \det \begin{pmatrix} -1-\lambda & 3 \\ 1 & -3-\lambda \end{pmatrix}$

$= (1+\lambda)(3+\lambda) - 3 = \lambda^2 + 4\lambda + \cancel{3} - \cancel{3} = \lambda^2 + 4\lambda$

$= \lambda(\lambda+4) \Rightarrow \lambda = \overset{\lambda_1}{0}, \overset{\lambda_2}{-4}$ are eigenvalues.

$\lambda_1 = 0 \Rightarrow (A - \lambda I)\vec{x} = 0$ is $\left(\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right) \to \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 3 \end{array}\right) \end{array}$

y free $\quad -x + 3y = 0 \Rightarrow x = 3y \quad \vec{x} = y\begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda_2 = -4 \quad (A - \lambda I)\vec{x} = 0 \Rightarrow \left(\begin{array}{cc|c} +3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right) \to \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}$

y free $\quad 3x + 3y = 0 \Rightarrow x = -y \quad \vec{x} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \vec{\xi}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\boxed{S = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}} \qquad S^{-1} = \dfrac{1}{1-(-3)}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} = \dfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$

Don't have to check $S^{-1}AS = \underline{\Lambda}$! where $\underline{\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$

b $e^{tA} = S e^{t\Lambda} S^{-1} = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-4t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}\frac{1}{4}$

$$e^{tA} = \begin{pmatrix} 1 & -e^{-4t} \\ 3 & e^{-4t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1+3e^{-4t} & 1-e^{-4t} \\ 3-3e^{-4t} & 3+e^{-4t} \end{pmatrix} = e^{tA}$$

(2)

~~(c)~~ (b)

Diff. eq. solution $\vec{x}(t) = e^{tA} \vec{x}(0)$

$$\vec{x}(t) = \cancel{\frac{1}{4}} \begin{pmatrix} 1+3e^{-4t} & 1-e^{-4t} \\ 3-3e^{-4t} & 3+e^{-4t} \end{pmatrix} \begin{pmatrix} \cancel{4}1 \\ \cancel{4}1 \end{pmatrix} = \begin{pmatrix} 2+2e^{-4t} \\ 6-2e^{-4t} \end{pmatrix}$$

$\vec{x}(t) =$

(c) as $t \to \infty$ $e^{-4t} \to 0 \Rightarrow \vec{x}(t) \to \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

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(2) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $p_{char}(\lambda) = \det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2$

$\Rightarrow$ $\boxed{\lambda = 2}$ is the only eigenvalue (repeated)

$$(A-\lambda I)\vec{x} = \vec{0} \Rightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}$$

x free
$y = 0$

$\vec{x} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

is the ONLY eigenvector!

You need 2 LINEARLY INDEPENDENT eigenvectors to form an S-matrix! $\Rightarrow$ cannot have an S here.

2b $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad e^{tB} = I + tB + \dfrac{t^2B^2}{2!} + \dfrac{t^3B^3}{3!} + \dots$

Here $B^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ all higher powers of B are also $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow e^{tB} = I + tB = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

They said $e^{tA} = e^{2t}e^{tB} = e^{2t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$

c $\dfrac{d}{dt}\left(e^{tA}\right) = \begin{pmatrix} 2e^{2t} & t\cdot 2e^{2t} + e^{2t} \\ 0 & 2e^{2t} \end{pmatrix}$ SAME!

$$Ae^{tA} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} & 2te^{2t} + e^{2t} \\ 0 & 2e^{2t} \end{pmatrix}$$

$$\Rightarrow \boxed{\frac{d}{dt}e^{tA} = Ae^{tA}}$$

3 $M = \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 5/12 & 5/12 \\ 1/6 & 5/12 & 5/12 \end{pmatrix}$ Let's start a & b together. We know a Markov matrix always has $\lambda_1 = 1$ as an eigenvalue. ALSO if 2 rows same $\Rightarrow \det(M) = 0$

$\Rightarrow \lambda_2 = 0$ is an eigenvalue.

(3a) For $\lambda_1 = 1$ they suggest $\vec{\xi} = (1,1,1)^T$ $\quad \vec{\xi}_1$

check $\begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 5/12 & 5/12 \\ 1/6 & 5/12 & 5/12 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 + 1/6 + 1/6 \\ 1/6 + 5/12 + 5/12 \\ 1/6 + 5/12 + 5/12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

YES!

(3b) For $\lambda_2 = 0$ $\quad M - \lambda_2 I = M$

$\Rightarrow \left(\begin{array}{ccc|c} 2/3 & 1/6 & 1/6 & 0 \\ 1/6 & 5/12 & 5/12 & 0 \\ 1/6 & 5/12 & 5/12 & 0 \end{array}\right) \xrightarrow[L_{31} = -1/4]{L_{21} = -\frac{1}{4}} \left(\begin{array}{ccc|c} 2/3 & 1/6 & 1/6 & 0 \\ 0 & 9/24 & 9/24 & 0 \\ 0 & 9/24 & 9/24 & 0 \end{array}\right)$

$L_{32} = 9/24$

$\rightarrow \left(\begin{array}{ccc|c} 2/3 & 1/6 & 1/6 & 0 \\ 0 & 9/24 & 9/24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

z free

$\frac{9}{24} y + \frac{9}{24} z = 0 \Rightarrow y = -z$

$\frac{2}{3} x + \cancel{\frac{1}{6}(-z)} + \cancel{\frac{1}{6}(z)} = 0$

$\Rightarrow x = 0$

$\vec{x} = z \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \vec{\xi}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

3c $\lambda_3 = 1/2 \quad (M - \lambda I)\vec{x} = \vec{0} \Rightarrow$

$$\left(\begin{array}{ccc|c} 1/6 & 1/6 & 1/6 & 0 \\ 1/6 & -1/12 & 5/12 & 0 \\ 1/6 & 5/12 & -1/12 & 0 \end{array}\right) \xrightarrow[L_{31}=1]{L_{21}=1} \left(\begin{array}{ccc|c} 1/6 & 1/6 & 1/6 & 0 \\ 0 & -5/12 & 5/12 & 0 \\ 0 & 5/12 & -5/12 & 0 \end{array}\right)$$

$$\xrightarrow{L_{32}=-1} \left(\begin{array}{ccc|c} 1/6 & 1/6 & 1/6 & 0 \\ 0 & -5/12 & 5/12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z free

$-\frac{5}{12}y + \frac{5}{12}z = 0 \quad y = z$

$\frac{1}{6}x + \frac{1}{6}(z) + \frac{1}{6}z = 0$

$x = -2z$

$$\vec{x} = z\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{\xi}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

3d YES! $\vec{\xi}_1^T \vec{\xi}_2 = 0 - 1 + 1 = 0$

$\vec{\xi}_1^T \vec{\xi}_3 = -2 + 1 + 1 = 0$

$\vec{\xi}_2^T \vec{\xi}_3 = 0 - 1 + 1 = 0$

SYMMETRY of M guaranteed ξ's ORTHOGONAL!

3e $Q = (\vec{q}_1\ \vec{q}_2\ \vec{q}_3)$ where each $\vec{q}_k = \dfrac{\vec{\xi}_k}{\|\vec{\xi}_k\|}$

$$\vec{q}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad \text{since } \|\vec{\xi}_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|\vec{\xi}_2\| = \sqrt{0+1+1} = \sqrt{2} \Rightarrow \vec{q}_2 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\|\vec{\xi}_3\| = \sqrt{4+1+1} = \sqrt{6} \quad \vec{q}_3 = \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

WE DID THIS IN CLASS in MARKOV MATRICES

3f Recall that "SUM of entries in each column = 1"

means $(1,1,1)M = (1,1,1)$, Therefore

$$[(1,1,1)M]^T = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ i.e. } M^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

if M SYMMETRIC

BUT $M^T = M \Rightarrow M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

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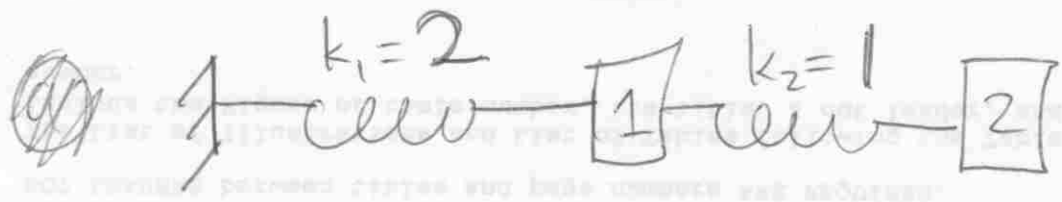
(a) 

This is the situation that I did not draw on the exam paper. There is NO support at the right; thats why the matrix looks so weird. Anyway

(a)

$$P_{char}(\lambda) = \det\begin{pmatrix} -3-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (3+\lambda)(1+\lambda) - 1$$

$$= \lambda^2 + 4\lambda + 2 \quad \text{does NOT factor!}$$

$$\Rightarrow \lambda = \frac{1}{2}\left[-4 \pm \sqrt{16 - 4\cdot 2}\right] = \frac{1}{2}\left[-4 \pm \sqrt{8}\right] = -2 \pm \sqrt{2}$$

BOTH NEGATIVE! Frequencies ω are $\sqrt{-\lambda}$

$$= \sqrt{-(-2 \pm \sqrt{2})} = \underbrace{\sqrt{2+\sqrt{2}}}_{\text{HIGH}}, \underbrace{\sqrt{2-\sqrt{2}}}_{\text{LOW}}$$

(b) Higher frequency corresponds to $\lambda = -2-\sqrt{2}$

$$(K-\lambda I)\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -1+\sqrt{2} & 1 & 0 \\ 1 & 1+\sqrt{2} & 0 \end{array}\right)$$

You can check that the lower row reduces to $(0\ 0 \mid 0)$

BUT it saves time to ASSUME it, which is OK provided you have the correct eigenvalue.

$$\Rightarrow \left(\begin{array}{cc|c} -1+\sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \begin{array}{c} y \text{ free} \\ (-1+\sqrt{2})x + y = 0 \end{array} \quad \Rightarrow x = \left(\frac{-1}{-1+\sqrt{2}}\right) y$$

$$\vec{x} = y \begin{pmatrix} -1/(\sqrt{2}-1) \\ 1 \end{pmatrix} \qquad \vec{\xi} = \begin{pmatrix} -1/(\sqrt{2}-1) \\ 1 \end{pmatrix}$$

We don't know much about $\sqrt{2}$ except that its between 1 & 2. Therefore x_1 component is <0 while x_2 component >0.

OPPOSITE DIRECTION MOTION at the same time!

Also $\sqrt{2}-1 < 1 \Rightarrow \frac{1}{\sqrt{2}-1} > 1 \Rightarrow$ mass 1 is moving MORE than mass 2.