Math 410 (Prof. Bayly) EXAM 3: Monday 12 April 2004

There are 4 problems on this exam. They are not all the same length, difficulty, or point value. You should read the exam through carefully before deciding which problem to work on first. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator. You may leave answers in terms of fractions or square roots of integers.

(1)(30 points) For the matrix $A = \begin{pmatrix} -1 & 7 \\ 1 & -7 \end{pmatrix}$,

(a)(15 points) Find its eigenvalues and eigenvectors, a matrix S and its inverse S^{-1} for which $S^{-1}AS$ would be diagonal (but you don't have to check this).

(b)(10 points) Find the matrix exponential e^{tA} and use it to solve the system of differential equations $d\vec{x}/dt = A\vec{x}$ with initial conditions $\vec{x}(t=0) = (4,4)^T$.

(c)(5 points) What is the limit of $\vec{x}(t)$ as $t \to \infty$?

(2)(20 points) For the matrix $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$,

(a)(5 points) Find the eigenvalue and eigenvector. Why is there no matrix S for which $S^{-1}AS$ is diagonal?

We can nevertheless find an expression for the matrix exponential e^{tA}, using the fact the A is the sum of $2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$. For matrices of this special form it is a fact that $e^{tA} = e^{2t}e^{tB}$.

(b)(5 points) Use the power series for the exponential function to calculate e^{tB}, and then write down the resulting formula for e^{tA}.

(c)(10 points) Differentiate your formula for e^{tA} (just differentiate each entry in the matrix), and check that the result is the same as multiplying e^{tA} by A [i.e. that $\frac{d}{dt}e^{tA} = Ae^{tA}$].

(3)(30 points) The matrix

$$M = \begin{pmatrix} 2/3 & 1/6 & 1/6 \\ 1/6 & 5/12 & 5/12 \\ 1/6 & 5/12 & 5/12 \end{pmatrix}$$

is symmetric and Markov (all entries nonnegative, sum of entries in each column =1).

(a)(5 points) The Markov property tells you the value of one eigenvalue - what is it? Verify that $(1,1,1)^T$ is an eigenvector of M belonging to this eigenvalue.

(b)(5 points) The second and third rows of M are the same, which tells you the value of another eigenvalue - find it and the corresponding eigenvector.

(c)(5 points) The third eigenvalue of M is 1/2; find the corresponding eigenvector.

(d)(5 points) Check that the eigenvectors belonging to the different eigenvalues are orthogonal. What property of M guaranteed this?

(e)(5 points) Find an orthonormal matrix Q for which $Q^T M Q$ has the eigenvalues of M on the diagonal and zero everywhere else.

(f)(5 points) Show that $(1,1,1)^T$ would be an eigenvector for any symmetric 3x3 Markov matrix, and say what eigenvalue it belongs to.

(4)(20 points) The positions $x_1(t), x_2(t)$ of a pair of unit masses connected by two springs satisfy $\frac{d^2}{dt^2}\vec{x} = K\vec{x}$, where

$$\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad , \quad K = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix}.$$

(a)(10 points) Find the eigenvalues of the matrix K. What are the natural frequencies of vibration?

(b)(10 points) For the higher frequency vibration, find the eigenvector. In this motion, are the masses moving in the same direction at the same time? Which mass is moving with greater amplitude?

You may leave your answers in terms of square roots.