(12) $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$

$B = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the nullspace of A

note $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = 0$ and $\begin{bmatrix} 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} = 0$

$x = (3,3,3)$, then $x = x_r + x_n$, $x_n = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$, $x_r = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

(8) $(1,1,0)$, $(1,0,1)$, $(0,1,1)$

center at $(1/2, 1/2, 1/2) = a$ what is angle between center and vertices

$a^T b / (\|a\| \|b\|)$

for any one of $(1,1,0)$, $(1,0,1)$, or $(0,1,1)$ $= b$

$a^T b = 1$, $\|a\| = \sqrt{3/4}$, $\|b\| = \sqrt{2}$

so $a^T b / (\|a\| \|b\|) = 1 \cdot 2/\sqrt{3} \cdot 1/\sqrt{2} = \sqrt{2}/\sqrt{3} = \cos\theta$

(13) $P = aa^T / a^T a$ $\qquad a = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

now $a^T a = \sum_{i=1}^{n} x_i^2$

now consider aa^T, $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1x_1 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & \ddots & & \\ \vdots & & \ddots & \\ x_nx_1 & & & x_nx_n \end{bmatrix}$

Then the entry of aa^T at i,j is $x_i x_j$

so the trace of aa^T is $\sum_{i=1}^{n} x_i^2$

so as long as $\sum_{i=1}^{n} x_i^2 \neq 0$, i.e. $a \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\left(\sum_{i=1}^{n} x_i^2\right) / \left(\sum_{i=1}^{n} x_i^2\right) = 1$$

(14) $E^2 = \|Ax - b\|^2$ $\qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad x = \begin{bmatrix} u \\ v \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$E^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$

$\partial E^2 / \partial u = 2u - 2 + 2u + 2(v-4)$

$\partial E^2 / \partial v = 2v + 2v - 6 + 2(u-4)$

set them equal to 0

$2u - 2 + 2u + 2v - 8 = 0 \longrightarrow 4u + 2v = 10 \rightarrow \boxed{2u + v = 5}$

$2v + 2v - 6 + 2u - 8 = 0 \qquad\;\; 4v + 2u = 14 \rightarrow \boxed{2v + u = 7}$

$A^T A x = A^T b$ $\qquad A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\qquad A^T b = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \rightarrow \boxed{\begin{array}{l} 2u + v = 5 \\ u + 2v = 7 \end{array}}$

5

$$Ax = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} = b$$

$$(C-D-4)^2 + (C-5)^2 + (C+D-9)^2 = f(C,D)$$

$$f(C,D) = 3C^2 - 4CD - 36C + 2D^2 + 26D + 122$$

$$\partial/\partial C \, f(C,D) = 6C - 4D - 36$$

$$\partial/\partial D \, f(C,D) = 4D - 4C + 26$$

so we have the following system

$$\left[\begin{array}{cc|c} 6 & -4 & 36 \\ -4 & 4 & -26 \end{array}\right]$$ which gives us $C = 5$, $D = -3/2$

$$P = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A(A^TA)^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1/3 & 0 \\ 1/3 & 1/2 \end{bmatrix}$$

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 1/3 & -1/2 \\ 1/3 & 0 \\ 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5/6 & 2/6 & -1/6 \\ 2/6 & 2/6 & 2/6 \\ -1/6 & 2/6 & 5/6 \end{bmatrix}$$

$$P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix} \qquad Pb = \begin{bmatrix} 7/2 \\ 6 \\ 17/2 \end{bmatrix}$$

⑥ $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

$P = A(A^TA)^{-1}A^T$

$A^TA = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix}$

$(A^TA)^{-1} = \begin{bmatrix} 2/9 & 1/9 \\ 1/9 & 1/9 \end{bmatrix}$ $A(A^TA)^{-1} = \begin{bmatrix} 3/9 & 4/9 \\ 3/9 & 1/9 \\ 0 & 2/9 \end{bmatrix}$

$1/9 \begin{bmatrix} 3 & 2 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} = 1/9 \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{bmatrix} = P$

$Pb = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$ $b - Pb = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

(14) $y = C + Dt + Ez$ with the four points

$y=3$ at $t=1, z=1$ $\quad$ $y=6$ at $t=0, z=3$

$y=5$ at $t=2, z=1$ $\quad$ $y=0$ at $t=0, z=0$

1)
$$\begin{aligned}3 &= C + D + E\\ 5 &= C + 2D + E\\ 6 &= C + 0D + 3E\\ 0 &= C + 0D + 0E\end{aligned}$$

2)
$$\begin{aligned}3 &= C + D + E\\ 5 &= C + 2D + E\\ 6 &= C + 0D + 3E\end{aligned}$$

4th equation will always contribute same error, regardless of C, D, and E

(23) given $y_1, \ldots y_m$ show the best least squares fit $y = C$

is $C = \dfrac{y_1 + \cdots y_m}{m}$

$$E(C) = (C - y_1)^2 + (C - y_2)^2 + \ldots + (C - y_m)^2$$

$$E'(C) = 2C - 2y_1 + 2C - 2y_1 + \ldots + 2C - 2y_m$$

$$E'(C) = 0$$

$$2mC + 2(y_1 + y_2 + y_3 + \ldots + y_m) = 0$$

$$C = \frac{y_1 + y_2 + \cdots y_m}{m}$$