5.3 5, 7, 8, 10, 16

5.4 4, 8, 13, 15, 20

5.3.5
$$\begin{bmatrix} d_{k+1} \\ s_{k+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} d_k \\ s_k \\ w_k \end{bmatrix}$$

$d_k + s_k/4 = d_k$ $\quad s_k = 0,\ w_k = 0,\ d_k = 0$ $\quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3s_k/4 + w_k/2 = s_k$

$w_k/2 = w_k$

5.3.7 find the limiting values of y_k and z_k and $A^k = S\Lambda^k S^{-1}$

$(k \to \infty)$ if $y_{k+1} = .8y_k + .3z_k$ $\quad y_0 = 0$

$z_{k+1} = .2y_k + .7z_k$ $\quad z_0 = 5$

$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ $\quad \lambda = 1, 1/2$

$\lambda = 1 \to (3/2, 1)$

$\lambda = 1/2 \to (-1, 1)$

$y_k = 3(1 - 1/2^k)$ $\quad z_k = 2 + 3/2^k$

$y_k \to 3 \quad z_k \to 2$

(8) $A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}$ $\quad \mathrm{Det}(A) = 0$

$\lambda_1 = 0 \quad x_1 = (-1, -1, 2)$

$\lambda_2 = 1,\ \lambda_3 = -1/5$

5.3.10 $A = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$ $\quad \lambda^2 + (b-a-1)\lambda + a - b = 0$

$\lambda = a-b,\ 1$ $\quad , u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} -b/(a+1) & -1 \\ 1 & 1 \end{bmatrix}$ $\quad u_k = S\Lambda S^{-1} u_0$ $\quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & a-b \end{bmatrix}$

a and b in $[0, 1]$

$$u_k = \begin{bmatrix} b/(1-a) & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (a-b)^k \end{bmatrix} \begin{bmatrix} b/(1-a) & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{b-a+1} \begin{bmatrix} 2b - (1-a-b)(a-b)^k \\ 2(1-a) - (1-a-b)(a-b)^k \end{bmatrix} \xrightarrow{\text{as}} \begin{bmatrix} 2b \\ 2(1-a) \end{bmatrix} \cdot 1/(b-a+1)$$

$, k \to \infty$

A is not necessarily a Markov Matrix. $a = -1/2,\ b = -1/2$

5.3.16 $(I-A)(I+A+A^2+\dots) = I$

$I + A + A^2 + A^3 + A^4 + \dots - (A + A^2 + A^3 + \dots) = I$

$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $n > 2$

so $(I + A + A^2 + \dots) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

5.4.4 if P is a projection matrix

$e^P = 1 + P + P^2/2! + P^3/3! + P^4/4! + \dots$

but $P^2 = P$ so $P^n = 1$

so $e^P = 1 + P(1 + 1/2! + 1/3! + 1/4! + \dots)$

$e^P = 1 + P(e-1)$

5.4.6 $\frac{dv}{dt} = 4v - 2w$ $\frac{dw}{dt} = v + w$

$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ $\lambda_1 = 3$, $\lambda_2 = 2$ $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 > 1$, $\lambda_2 > 1$ unstable

b) $100e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 100e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, ratio $\to 2$

5.4.13 $\frac{du}{dt} = Au = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$u_1' = cu_2 - bu_3$, $u_2' = -cu_1 + au_3$, $u_3' = bu_1 - au_2$

$u_1' \cdot u_1 + u_2' u_2 + u_3' u_3 = u_1(cu_2 - bu_3) + u_2(-cu_1 + au_3) + u_3(bu_1 - au_2) = 0$

b) $\|u(t)\|^2 = \|e^{At} u_0\|^2 = \|u_0\|^2$ = constant

c) $\lambda_1 = 0$, $\lambda_{2,3} = \pm i (a^2 + b^2 + c^2)^{1/2}$

5.4.15 $\frac{d^2u}{dt^2} = \begin{bmatrix} -5 & -1 \\ -1 & -5 \end{bmatrix} u$ with $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_0' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = -4, -6$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $a_1 + a_2 = 0$, $a_2 - a_1 = 1$, $a_2 = a_1 = 1/2$

$$u(t) = \frac{1}{2} \cos 2t \begin{bmatrix} 1 \\ .1 \end{bmatrix} + \frac{1}{2} \cos \sqrt{6} t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5.4.20 $\frac{du}{dt} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$ $u_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$ $\lambda = 1, 3, 4$

$$u = \begin{bmatrix} 8/3 e^t - 6e^{3t} + 13/3 e^{4t} \\ -6e^{3t} + 6e^{4t} \\ e^{4t} \end{bmatrix}$$