Math 410 (Prof. Bayly) CURRENCY ARBITRAGE Summer 2005

One of the tragedies of the 21st century is the replacement of the distinctive European currencies (francs, marks, lire, gelder, kroner etc.) by the single Euro. Well, it does simplify international commerce, but some people regret the loss of national character embodied in the old currencies. There was a time in the early days of the USA when each state issued its own currency, and there were probably people who were saddened to see the advent of a single national currency.

There is a special sadness experienced by the currency speculators, people who make their livings by trading currencies into each other, carefully studying the fluctuations so that the average result of all their trades is a net increase in their cash holdings†. The European currency market was rich and varied in the opportunities it presented, and the sheer amount of money changing form on a regular basis made it possible for a large number of people to make a lot of money‡ Fortunately - from the speculators' point of view - the world is far from having a single currency, and there will continue to be opportunities for making money in this field.

Now speculation usually involves predicting that the relative values of two currencies will be different at a later time than they are now, and buying or selling accordingly. Arbitrage is somewhat different; arbitrage involves considering all the exchange rates at a given moment and conducting a number of transactions at the given rates, whose effect is to increase the total amount of money. Speculation involves risk; arbitrage has no risk, but on the other hand arbitrage is not always possible.

Example 1: 2 country arbitrage. Imagine two countries A and B whose currencies are shillings and florins, and two banks a and b that exchange shillings for florins and vice versa. Bank a considers 3 florins equivalent to 2 shillings (i.e. 3/2 florin per shilling), while bank b considers 4 florins equivalent to 3 shillings (i.e. 4/3 florin per shilling). We can diagram this (figure 1a) using a digraph with two edges, each representing on of the banks. If you start with one shilling, turn it into florins with bank a and then back into shillings with bank b, you will end up with 9/8 shillings. In other words, you have made yourself 1/8 shilling with no work at all, and no risk. Arbitrage is the term that denotes such an

† Speculation should be distinguished from the practice of simply charging a percentage for the service of changing money from one currency to another.

‡ From whom do speculators make their money? From people who do NOT have a choice about what currency to change into what other currency, or when they do it. Tourists with set vacation plans, students whose colleges are not in their home countries, businesses that operate across national boundaries, these folks need to change one specific currency into another, and they need it at a specific time. They are willing (or compelled) to lose a little in order in the process. It is probably - not to go too deeply into financial issues I have little understanding of - the demands from people who *need* a certain transaction at a certain time that drive the fluctuations in currency values that the speculators live off.

opportunity to make money with no work and no risk, due solely to a price discrepancy.

Well, that's unlikely to happen in practice - one bank notices pretty fast if another bank is exchanging the same currencies at a different rate. So all banks that exchange currencies A and B use the same exchange rate, and we only need one edge linking countries A and B. We're assuming the banks do not charge fees or percentages; they purely convert money from one form to another at the prescribed rate. Real banks charge a fee or percentage for the service of exchanging currencies. To make money, the increase in value has to exceed the transaction fees.

Example 2: 3 country arbitrage. Let's begin by supposing the existing banks have agreed that 3 shillings are equivalent to 4 florins, so edge a corresponds to a factor 4/3. Now suppose country C enters the international marketplace with its currency guilders. Exchanges between A and C occur at rate 3/5 guilder per shilling (therefore 5/3 shilling per guilder), denoted by edge c, and between B and C at rate 2 florin per guilder (i.e. 1/2 guilder per florin), denoted by edge d. We have already prevented 2-country arbitrage opportunities from happening, but you can quickly calculate that if you start with 1 shilling, you can change it forwards along a into 4/3 florin, then backwards along d to 2/3 guilder, then forwards along b to 10/9 shilling. You have made 1/9 of a shilling with no work at all, and no risk.

This type of arbitrage opportunity might be harder to identify than between only two countries, because it arises indirectly through a third country and two other banks, which might not have much information about what each other are doing. Presumably once somebody starts taking advantage of the arbitrage opportunity, the individual bankers will notice the activity and move to adjust their exchange rates. A practical question is how fast international bankers can adjust exchange rates relative to the time it takes an arbitrageur to execute a profitable cycle of exchanges.

General picture. You probably see where we're headed! An international currency market consists of a network of countries joined by banks that exchange currencies between the countries. Each edge is characterized by a rate for exchange in the indicated direction, with the rate for the opposite exchange given by the inverse. The question is whether we can find a loop in the network, such that if we travel around the loop exchanging money at the given rates, can we return to the starting point with more than we started with?

From a mathematical point of view, the main challenge is finding the loops! However, if we have the edge-node incidence matrix G, then basic loops in the network correspond to basis vectors for the nullspace of G^T. Since all loops are combinations of the basic loops, once we have found a basis for the nullspace of G^T, we can just check each basic loop and see if the product of exchange factors around the loop is different from unity. If so, then there is an arbitrage opportunity. Strictly speaking, we want the product of exchange factors to be greater than unity in order to make money. However, if the product of exchange factors is less than unity going around a loop in one direction, then simply going around the loop in the opposite direction will give a product greater than unity.

Kirchoff's Currency Exchange Law. The problem of identifying arbitrage opportunities is now solved, at least in this simple model. But our inner nine-year-old is complaining "Multiplication is *hard*! Can't we do addition and subtraction?" The answer is yes, at the price of a little transcendentality; multiplying the exchange factors around a loop is equivalent to adding the logarithms of said exchange factors. And the criterion that the product of exchange factors be unity or not is the same as seeing whether the sum of logs of exchange factors is zero or nonzero. (It does not matter what base we're using for our logarithms! Popular choices are 2, e, and 10, but any real number greater than 1 works.)

We have seen sums around loops before, specifically Kirchoff's Voltage Law that says the sum of voltage drops around any loop in an electrical circuit must be zero. So we may expect an analogy between electrical circuits and international currency markets, with the logarithms of exchange rates corresponding to voltage drops. The precise analogy is that if the currency market "circuit" satisfies Kirchoff's law, there is no arbitrage opportunity; conversely if Kirchoff's law is violated around some loop in the circuit, then there is an arbitrage opportunity around that loop.

Financial Voltage? The electrical analogy is precise when the currency market circuit satisfies Kirchoff's voltage law. Now, you may recall that the practical consequence of the sum of voltage drops being equal to zero around all loops, is that voltage itself can be defined unambiguously at all nodes of the circuit. Well, we can define the voltage relative to an arbitrary "ground" potential - you recall that only voltage *differences* are physically important.

In the currency situation, if Kirchoff's law holds, there must exist a quantity like voltage (call it *financial voltage*) whose difference between adjacent nodes equals the logarithm of the exchange rate on the edge. Its interpretation, which makes sense when you think about it for a while, is as follows. If we choose one currency arbitrarily to be a reference, then each other currency - whether directly connected to the reference or not - can be unambiguously related to the reference currency with a well-defined exchange rate. That exchange rate is given by the exponential (using the same base as the logarithms) of the difference in financial voltage between the currency of interest and the reference currency. In fact, every exchange rate is the exponential of the financial voltage difference between the currencies involved.

In short, a globally consistent set of currency values exists if and only if Kirchoff's Voltage Law (applied to the logarithms of individual exchange rates) is satisfied. Arbitrage opportunities arise only if KVL is not satisfied, in which case there is no globally consistent set of currency values. Which makes sense, because the whole idea of arbitrage is somehow make a small amount of money in some currency equivalent - via exchanges - to a large amount of money in the same currency!

Critique. Casting a quick eye over the Wall Street Journal, I get the impression that the US dollar serves as a fairly good approximation to a global reference currency, which implies that to a fairly good approximation there are no arbitrage opportunities. So if you want to make money this way, you will be working with profits that are relatively small compared to the amounts of money you're exchanging. In my examples above, the factors were 9/8 and 10/9; I expect the factors in practice to be a lot closer to unity. One strategy to increase your profits, therefore, is to work with a lot of money - preferably someone else's! Join a bank or investment house and see if this method makes them (and you) rich. If it does, please make me rich.

Actually you can identify the potential to make money just by getting hold of data - what ARE the exchange rates that banks are actually using? You probably can't get it just from the Wall Street Journal, because that only gives rates to exchange foreign currencies for dollars. See if you can tap into the exchange rates offered by big banks in Europe, Japan, Mexico, Brazil, India etc., and see how closely Kirchoff's Voltage Law is satisfied (or how far it's off).

A major issue that has to be addressed in practice is those service fees or percentages I assumed not to exist. No bank is going to let money go through its hands with sticking to some of it - that is how they stay in business. If you want to make money, your arbitrage profits have to outweigh the losses to the banks. How you handle it depends on how the modified exchange rates are presented. A simple modification would be to state a single exchange rate, and then charge either a set fee or a percentage of the amount exchanged, regardless of the direction of the exchange. In that situation we could still characterize the bank by a single edge, with the exchange rate and fee given.

A more complicated situation (which seems to be more common in practice) is for banks to advertise two rates for each pair of currencies, so that the product of the two rates is slightly smaller than unity. In that case I suspect there is no alternative but to represent each direction with its own edge. The resulting theory in terms of graphs and loops and Kirchoff laws would be similar, except that the edges would have to be considered diode-like, allowing current to flow only in the indicated direction. Mathematically, the algebra of loops would be complicated by the requirement of considering null vectors of G^T whose entries are all nonnegative. Aargh!

FINAL COMMENT: We have seen Kirchoff's Voltage Law in three contexts now: the original situation of electrical circuits, the sports rankings problem, and now currency arbitrage. Not only are the contexts different, but the interpretations very different. In the circuits, KVL is a law of Nature; it is always satisfied. In the sports rankings we hope that KVL is somewhat close to being satisfied, but we expect it to be violated and are prepared to deal with the consequences. In the international currency market, we hope it is violated for then we can make some fast cash!

The idea that ties these, and probably many other applications, together is that KVL is a solvability condition for the associated system of equations. In these situations, there is, or may be, a global quantity like voltage or athletic quality that relates everything to everything else. Whether this quantity exists, and its properties, depends on the solvability condition.

That's WHY Prof Bayly makes such a big deal about the basis vectors for the null space of G^T!