$\cancel{3}k > 16\left(\frac{1}{11}+1=\frac{12}{11}\right) = \frac{16\times\cancel{12}\,4}{11}$ so $\boxed{k > \frac{16\times 4}{11} = \frac{64}{11}}$

slightly $> 16/3$

Math 410 (Prof. Bayly/Foth) FINAL EXAM: Wednesday 10 August 2005

There are 10 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can.

You will not need a calculator on this exam. If your calculations become numerically awkward and time-consuming, you may describe the steps you would take if you had a calculator.

It is EXTREMELY important to show your work! Correct answers without documented support will have points deducted.

You can also do this problem by determinants.

(1)(10 points) For what values of k is the quadratic form $3x^2+ky^2-8xy+xz+z^2$ positive definite?

$q(\vec{x}) = \vec{x}^T A \vec{x}$ where $A = \begin{pmatrix} 3 & -4 & 1/2 \\ -4 & k & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$

To tell pos-def, row-reduce

$A \xrightarrow[L_{31}=1/6]{L_{21}=-4/3} \begin{pmatrix} 3 & -4 & 1/2 \\ 0 & k-\frac{16}{3} & 2/3 \\ 0 & 2/3 & 11/12 \end{pmatrix} \xrightarrow{L_{32}=\frac{2/3}{k-16/3}=\frac{2}{3k-16}}$

$\begin{pmatrix} 3 & -4 & 1/2 \\ 0 & k-16/3 & 2/3 \\ 0 & 0 & \frac{11}{12}-\frac{2}{(3k-16)}\cdot\frac{2}{3} \end{pmatrix}$

pos DEF if $k > 16/3$

AND $\frac{11}{12} > \frac{4/3}{3k-16}$

i.e. $3k-16 > \frac{48}{33}$

$3k > \frac{\cancel{48}\,16}{\cancel{33}\,11} + 16$

(2)(10 points) Find the closest point on the plane spanned by vectors $\mathbf{v}_1 = (1, 0, 1)^T$ and $\mathbf{v}_2 = (0, 1, 2)^T$ to the point $P = (3, 2, 1)$. Also compute the shortest distance.

I.e. Find x, y so that $x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}0\\1\\2\end{pmatrix}$ is as close as possible to $\begin{pmatrix}3\\2\\1\end{pmatrix}$

I.e. try to solve $\underbrace{\begin{pmatrix}1 & 0\\0 & 1\\1 & 2\end{pmatrix}}_{A}\begin{pmatrix}x\\y\end{pmatrix} = \underbrace{\begin{pmatrix}3\\2\\1\end{pmatrix}}_{\vec{b}}$ approximately, by least squares:

$A^TA\vec{x} = A^T\vec{b}$ Here $A^TA = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}2 & 2\\2 & 5\end{pmatrix}$

$A^T\vec{b} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix}$

$\left(\begin{array}{cc|c}2 & 2 & 4\\2 & 5 & 4\end{array}\right) \xrightarrow{L_{21}} \left(\begin{array}{cc|c}2 & 2 & 4\\0 & 1 & 0\end{array}\right)$ $y = 0$, $2x = 4$, $x = 2$ So $A\vec{x} = \begin{pmatrix}1 & 0\\0 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}$

$= \begin{pmatrix}2\\0\\2\end{pmatrix}$ is closest point to $\begin{pmatrix}3\\2\\1\end{pmatrix}$ in the given plane.

(3)(20 points) The linear system $A\vec{x} = \vec{b}$ has an infinite number of solutions with one free variable, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

(a)(5 points) Find the general solution and identify the free variable and null vector.

(b)(5 points) Express the squared length of the solution as a function of the free variable, and find the solution with the minimum length.

(c)(5 points) Calculate $K = AA^T$ and solve $K\vec{u} = \vec{b}$ for $\vec{u}$.

(d)(5 points) Calculate $A^T\vec{u}$. Is it the same as the min length solution you found in (b)?

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$ is already in echelon form! / Reduced also!

$\Rightarrow \boxed{z \text{ free}}$ $\quad y = z+2$, $x = 1-2z$ $\Rightarrow \vec{x} = \begin{pmatrix} 1-2z \\ z+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

null vector $\vec{z}$

(b) $L^2(z) = (1-2z)^2 + (z+2)^2 + z^2$

$= (4z^2 - \cancel{4z} + 1) + (z^2 + \cancel{4z} + 4) + z^2 = 6z^2 + 5$

To minimize, use $L^{2\prime}(z) = 12z = 0$ when $z=0$!

$\Rightarrow \vec{x}_{ml} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

(c) $AA^T = \begin{pmatrix}1&0&2\\0&1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\2&-1\end{pmatrix} = \begin{pmatrix}5&-2\\-2&2\end{pmatrix}$

$\left(\begin{array}{cc|c}5&-2&1\\-2&2&2\end{array}\right) \xrightarrow{l_{21}=\frac{-2}{5}} \left(\begin{array}{cc|c}5&-2&1\\0&6/5&18/5\end{array}\right) \Rightarrow v = \cancel{4/3}\ 2$

$5u - \cancel{8/6}\,4 = 1 \qquad 5u = 5 \quad u = 1$

$\vec{u} = \begin{pmatrix}1\\2\end{pmatrix}$

$A^T\vec{u} = \begin{pmatrix}1&0\\0&1\\2&-1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix}$

(4)(20 points) Consider the symmetric matrix $A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$.

(a)(10 points) Find the eigenvalues and eigenvectors of A, and verify that the eigenvectors are orthogonal.

(b)(10 points) Find a matrix Q whose columns are orthonormal vectors, for which $Q^T A Q = \Lambda$, a diagonal matrix. Verify by direct calculation of $Q^T A Q$.

a Evalues $\det(A-\lambda I)=0$ $\quad \det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix}$

$= \lambda^2 - 7\lambda + 10 - 4 = \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6)$

$\Rightarrow \lambda = 1, 6$

$\lambda_1 = 1 \quad (A-\lambda_1 I)\vec{x} = \vec{0} \quad \left(\begin{array}{cc|c} 1 & 2 & 0 \\ \cancel{2} & \cancel{4} & \cancel{0} \end{array}\right) \to 0's$ $\quad$ y free $\; x+2y=0 \quad \vec{x} = \begin{pmatrix} -2y \\ y \end{pmatrix}$

$\Rightarrow \vec{\xi}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda_2 = 6 \quad (A-\lambda_2 I)\vec{x} = \vec{0} \quad \left(\begin{array}{cc|c} -4 & 2 & 0 \\ \cancel{2} & \cancel{-1} & \cancel{0} \end{array}\right) \to 0's$ $\quad$ y free $\; -4x+2y=0 \quad x = \frac{1}{2}y$

$\vec{x} = \begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix} = y\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ if $y=2$ $\Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ortho? $\langle \vec{\xi}_1, \vec{\xi}_2 \rangle = -2\cdot 1 + 1\cdot 2 = 0$! ✓

$\Rightarrow \perp$

(b) Choose Q to have NORMALIZED eigenvectors as columns

$$\vec{q}_1 = \frac{\vec{\xi}_1}{\|\vec{\xi}_1\|} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Big/ \sqrt{5} \quad , \quad \vec{q}_2 = \frac{\vec{\xi}_2}{\|\vec{\xi}_2\|} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Big/ \sqrt{5}$$

$$\|\vec{\xi}_1\| = \sqrt{1+4} = \sqrt{5}$$

$$\Rightarrow \boxed{Q = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}}$$

Note $Q^T = Q$!

Verify
$$Q^T A Q = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \frac{1}{\sqrt{5}}$$

$$= \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ 1 & 12 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 30 \end{pmatrix} = \begin{pmatrix} \overset{=\lambda_1}{1} & 0 \\ 0 & \underset{=\lambda_2}{6} \end{pmatrix} = \Lambda \, !$$

(5)(10 points) The *Goose and Gherkin* and *No Octopi* are neighboring restaurants that start the year with 75 customers each. The Goose regularly presents live music, with the result that 80 per cent of the patrons one night return on the next night, with the other 20 per cent going to No Octopi for some quiet pizza. Meanwhile 60 per cent of the customers at No Octopi return the next night, with 40 per cent going over to the Goose.

As weeks and weeks go by (i.e. as time goes to infinity), what are the expected numbers of customers at the two restaurants?

Transition matrix $T = \begin{array}{c c} & \begin{array}{cc} G & N \end{array} \\ \begin{array}{c} G \\ N \end{array} & \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix} \end{array}$ is a REGULAR trans. matrix

KNOW $\lambda = 1$ is the only important eigenvalue in the long term, and its eigenvector gives long-term distribution:

$(T - 1I)\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -.2 & .4 & 0 \\ \cancel{.2} & \cancel{.4} & \cancel{0} \end{array}\right) \to 0\text{s}$ y free $\quad -.2x + .4y = 0 \quad x = 2y$

$\Rightarrow \vec{x} = y\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Choose y so that TOTAL population = Total starting population $= 150$

$2y + y = 150 \qquad 3y = 150 \qquad y = 50$

$x = 2y = 100$

$\Rightarrow$ EXPECT $\begin{pmatrix} 100 \\ 50 \end{pmatrix}$ on average at $\begin{pmatrix} \text{GOOSE} \\ \text{NO OCTOPI} \end{pmatrix}$

(6)(20 points) Three soccer teams play three games over two days, and we want to rank them. In this problem you are encouraged to NOT solve any linear systems if you can answer the questions otherwise.

In case you've forgotten, we represent a game by an arrow from the "visitor" to the "home" team, and label with the score difference (home) - (visitor).

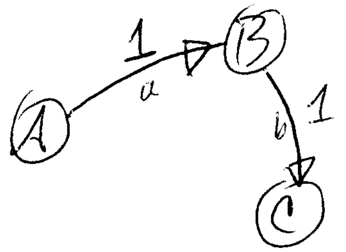
(a)(5 points) On the first day A visits B and loses by 1 goal, and B visits C and also loses by 1 goal. Draw a digraph that expresses this situation, and rank the teams, if possible.

(b)(5 points) On the second day, A visits C and wins by 4. Draw the digraph, and say why there is no perfectly consistent ranking.

(c)(5 points) Write down the edge-node matrix A for the digraph in (b), and find (without doing any calculation) a kernel vector $\vec{s}$ of A^T.

(d)(5 points) Find a linear system of equations whose solution would give a reasonable approximate ranking. You do NOT have to solve them, but you should find the matrix and right-hand side vector explicitly.

(a)



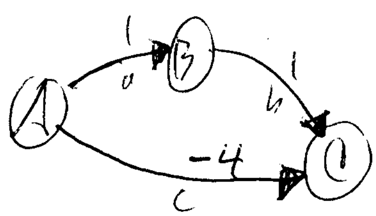
C is best, then B, then A

B is 1 better than A

C is 1 better than B

= 2 better than A

(b)



This would imply A is 4 better than C which is inconsistent with ranking in (a)!

(c)

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \end{array}$$

kernel of A^T corresponds to LOOPS.

There is only one loop in this graph, $A \to B \to C$ which has edge components

+1 along a, +1 along b, -1 along c $\Rightarrow \vec{s} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Check $A^T\vec{s} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓

(d) We would like a solution to $A\vec{Q} = \vec{D}$

where $\vec{D} = \begin{pmatrix}1\\1\\-4\end{pmatrix}$ is the score-difference vector.

BUT since inconsistent, we can only get Least Squares

$A^TA\vec{Q} = A^T\vec{D}$. Here $A^T\vec{D} = \begin{pmatrix}-1&0&-1\\1&-1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1\\1\\-4\end{pmatrix} = \begin{pmatrix}3\\0\\-3\end{pmatrix}$

$$A^TA = \begin{pmatrix}-1&0&-1\\1&-1&0\\0&1&1\end{pmatrix}\begin{pmatrix}-1&1&0\\0&-1&1\\-1&0&1\end{pmatrix} = \begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$$

Don't solve! But if we did....

$$\left(\begin{array}{ccc|c}2&-1&-1&3\\-1&2&-1&0\\-1&-1&2&-3\end{array}\right) \to \left(\begin{array}{ccc|c}2&-1&-1&3\\0&+3/2&-3/2&3/2\\ \cancel{0}&\cancel{-3/2}&\cancel{3/2}&\cancel{-3/2}\end{array}\right) \to 0\text{'s}$$

z free

$y - z = 1 \Rightarrow y = z+1$

$2x - (z+1) - z = 3 \qquad 2x = 2z+4 \qquad x = z+2$

$$\Rightarrow \vec{x} = \vec{Q} = \begin{pmatrix}z+2\\z+1\\z\end{pmatrix} = z\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}Q_A\\Q_B\\Q_C\end{pmatrix} \Rightarrow$$

A best
B next
C last!

(7)(10 points) Construct polynomials P_0, P_1, and P_2 of degree 0, 1, and 2 respectively, which are orthogonal with respect to the inner product $\int_0^1 f(t)g(t)t\,dt$.

Lets start with ~~[illegible]~~ $f_0(t) = 1$

$f_1(t) = t$, $f_2(t) = t^2$

Gram Schmidt.

$$p_0(t) = f_0(t)$$

$$p_1(t) = f_1(t) - \frac{\langle p_0, f_1\rangle}{\langle p_0, p_0\rangle} p_0(t)$$

where $$p_2(t) = f_2(t) - \frac{\langle p_0, f_2\rangle}{\langle p_0, p_0\rangle} p_0(t) - \frac{\langle p_1, f_2\rangle}{\langle p_1, p_1\rangle} p_1(t)$$

Here $p_0(t) = f_0(t) = 1$

$$\langle p_0, p_0\rangle = \int_0^1 1\cdot 1\, t\,dt = \left[\frac{t^2}{2}\right]_0^1 = 1/2$$

$$\langle p_0, f_1\rangle = \int_0^1 1\cdot t\cdot t\,dt = \left[\frac{t^3}{3}\right]_0^1 = 1/3$$

$$\Rightarrow p_1(t) = t - \frac{1/3}{1/2} 1 = t - 2/3 = p_1(t)$$

$$\langle p_1, p_1\rangle = \int_0^1 (t-2/3)^2 \, t\,dt = \int_0^1 \left(t^2 - \tfrac{4}{3}t + \tfrac{4}{9}\right) t\,dt$$

$$= \cancel{\left[\tfrac{t^3}{3} - \tfrac{2}{3}t^2\right.} \int_0^1 \left(t^3 - \tfrac{4}{3}t^2 + \tfrac{4}{9}t\right) dt$$

$$= \left[\frac{t^4}{4} - \frac{4t^3}{9} + \frac{2}{9}t^2\right]_0^1 = \frac{1}{4} - \frac{4}{9} + \frac{2}{9} = \frac{9-16+8}{36} = \frac{1}{36}$$

$$\langle p_1, f_2\rangle = \int_0^1 (t-2/3)\, t^2 \, t\,dt = \int_0^1 (t-2/3)\, t^3\,dt$$

$$= \int_0^1 \left(t^4 - \tfrac{2}{3}t^3\right) dt = \left[\frac{t^5}{5} - \frac{t^4}{6}\right]_0^1 = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$$

~~So~~ $\cancel{p_2(t)} =$ Also $\langle p_0\, f_2\rangle = \int_0^1 1\, t^2\, t\,dt = \int_0^1 t^3\,dt$

$$= \left[t^4/4\right]_0^1 = 1/4$$

So $p_2(t) = t^2 - \dfrac{1/4}{1/2}(1) - \dfrac{1/30}{1/36}(t-2/3)$

$$p_2(t) = t^2 - \frac{6}{5}t + \underbrace{\frac{\cancel{6}^{2}}{5}\cdot\frac{2}{\cancel{3}} - \frac{1}{2}}_{\cancel{\frac{24}{30}}\ \frac{8}{10} - \frac{5}{10} = \frac{3}{10}} = \boxed{t^2 - \frac{6}{5}t + \frac{3}{10} = p_2(t)}$$