MATH 223, Linear Algebra
Fall 2007
Midterm exam, Wednesday, October 17, 2007

Instructions:

• No notes, books or calculators permitted.

• This exam has six questions. All questions are worth the same number of marks.

• Do all of your work on the sheets provided. Do not separate sheets that have been stapled together. If you need more space for a question, use the pages at the end of each part; please indicate on those pages which question you are continuing.

• The questions have been divided into two parts, purely to facilitate marking. Make sure you have a “white” and a “blue” set of questions. Make sure that your name, student number, and section number are on both parts. (If your instructor is Bryden Cais, you are in section 1; if your instructor is Jim Loveys, you are in section 2.)

• Have fun!
PART 1.

. Name (Please PRINT clearly):

. Student number:

. Section number:
1. (a) Solve the system of linear equations over the complex numbers

\[
\begin{align*}
    x_1 + (2 + i)x_2 &= 7 - 3i \\
    (3 + i)x_1 + (6 + 6i)x_2 &= -2 + 8i
\end{align*}
\]

(b) Express the matrix

\[
A = \begin{pmatrix}
    1 & 2 + i \\
    3 + i & 6 + 6i
\end{pmatrix}
\]

as a product of elementary matrices.

(c) Find the inverse of the matrix \(A\) from part (b).
2. Let \( V = P(t) \) be the real vector space of polynomials with real coefficients. For each of the following subsets of \( P(t) \), decide whether it is or is not a subspace of \( V \). Justify your answers.

(a) \( S_1 = \{ p \in V \mid p(7) = 0 \} \).

(b) \( S_2 = \{ p \in V \mid p(0) = 7 \text{ or } p \text{ is the zero polynomial} \} \).

(c) \( S_3 = \{ p \in V \mid p \text{ is odd} \} \). [N.B. an odd polynomial \( p(t) \) is one that satisfies \( p(-t) = -p(t) \) for every real number \( t \).]
3. The following matrix is over the reals. Find a basis for its row space, its column space, and its null space.

\[
\begin{pmatrix}
1 & -3 & 4 & 0 & 3 \\
3 & -9 & 11 & 6 & 8 \\
-2 & 6 & -7 & -6 & -5 \\
\end{pmatrix}
\]
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PART 2.

. Name (Please PRINT clearly):

. Student number:

. Section number:
4. Let $W_1$ and $W_2$ be the subspaces of $\mathbb{R}^4$ defined by

\[ W_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad W_2 = \text{span} \left\{ \begin{pmatrix} 4 \\ 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ 2 \end{pmatrix} \right\}. \]

(a) Find a basis for $W_1 + W_2$.
(b) Find a basis for $W_1 \cap W_2$.
(c) Compute $\dim(W_1 + W_2)$ and $\dim(W_1 \cap W_2)$. 
5. Suppose that $A$ is a fixed $n \times n$ matrix, $V = M_n(F)$ is the vector space of $n \times n$ matrices over the field $F$, and $T : V \rightarrow V$ is the following function.

$$T(X) = AX -XA \quad \text{for each } X \in V.$$ 

(a) Show that $T$ is a linear operator on $V$ (i.e., that $T : V \rightarrow V$ is a linear mapping).

(b) Now suppose that $n = 2$ and that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$ 

Let $B$ be the standard ordered basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $V$. Find $[T]_B$.

(c) Find a basis for each of $\ker(T)$ and $\text{im}(T)$ (using the matrix $A$ from part (b)).
6. Suppose that $A$ is an $n \times n$ matrix over a field $F$. Show that the following conditions on the matrix $A$ are equivalent.

(a) $A$ is invertible.
(b) For every $n \times n$ matrix $B$ over $F$, there is a solution to the matrix equation $AX = B$.
(c) For every $n \times n$ matrix $B$ over $F$, there is a unique solution to the matrix equation $AX = B$. 
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