1. (a) Solve the system of linear equations over the complex numbers

\[
\begin{align*}
x_1 + (2 + i)x_2 &= 7 - 3i \\
(3 + i)x_1 + (6 + 6i)x_2 &= -2 + 8i
\end{align*}
\]

(b) Express the matrix

\[
A = \begin{pmatrix} 1 & 2 + i \\ 3 + i & 6 + 6i \end{pmatrix}
\]

as a product of elementary matrices.

(c) Find the inverse of the matrix \( A \) from part (b).

**Solution:** We will solve all three parts simultaneously by row-reducing the doubly-augmented matrix

\[
M := \begin{pmatrix} 1 & 2 + i & 7 - 3i & 1 & 0 \\ 3 + i & 6 + 6i & -2 + 8i & 0 & 1 \end{pmatrix}
\]

We obtain:

\[
M \xrightarrow{R_2-(3+i)R_1-R_2} \begin{pmatrix} 1 & 2 + i & 7 - 3i & 1 & 0 \\ 0 & 1 + i & -26 + 10i & -3 - i & 1 \end{pmatrix}
\]

\[
\xrightarrow{(\frac{1}{1+i})R_2-R_2} \begin{pmatrix} 1 & 2 + i & 7 - 3i & 1 & 0 \\ 0 & 1 & -8 + 18i & -2 + i & \frac{1-i}{2} \end{pmatrix}
\]

\[
\xrightarrow{R_1-(2+i)R_2-R_1} \begin{pmatrix} 1 & 0 & 41 - 31i & 6 & \frac{-3+i}{2} \\ 0 & 1 & -8 + 18i & -2 + i & \frac{1-i}{2} \end{pmatrix}
\]

(a) From our work above, we read off the unique solution \( x_1 = 41 - 31i \) and \( x_2 = -8 + 18i \).

(b) Note that many solutions are possible; we present one. Denoting by \( e_1, e_2, e_3 \) the three elementary row operations above, we have \( e_3(e_2(e_1(A))) = I_2 \), or what is the same thing

\[
A = e_1^{-1}(e_2^{-1}(e_3^{-1}(I))) = E'_1E'_2E'_3,
\]
where \( E'_i := e_i^{-1}(I) \). From this definition of \( E'_i \), we readily compute

\[
E'_1 = \begin{pmatrix} 1 & 0 \\ 3 + i & 1 \end{pmatrix} \quad E'_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 + i \end{pmatrix} \quad E'_3 = \begin{pmatrix} 1 & 2 + i \\ 0 & 1 \end{pmatrix}
\]

and so

\[
A = \begin{pmatrix} 1 & 0 \\ 3 + i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 + i \end{pmatrix} \begin{pmatrix} 1 & 2 + i \\ 0 & 1 \end{pmatrix}.
\]

(c) We read off the inverse from our initial work:

\[
A^{-1} = \begin{pmatrix} 6 & \frac{-3+i}{2} \\ -2 + i & \frac{1-i}{2} \end{pmatrix}.
\]

2. Let \( V = P(t) \) be the real vector space of polynomials with real coefficients. For each of the following subsets of \( P(t) \), decide whether it is or is not a subspace of \( V \). Justify your answers.

(a) \( S_1 = \{ p \in V \mid p(7) = 0 \} \).

(b) \( S_2 = \{ p \in V \mid p(0) = 7 \text{ or } p \text{ is the zero polynomial} \} \).

(c) \( S_3 = \{ p \in V \mid p \text{ is odd} \} \). [N.B. an odd polynomial \( p(t) \) is one that satisfies \( p(-t) = -p(t) \) for every real number \( t \).]

Solution:

(a) We claim that \( S_1 \) is a subspace. Indeed, \( 0 \in S_1 \) since the zero polynomial has value 0 everywhere (so in particular at 7). If \( p, q \in S_1 \) and \( k \in \mathbb{R} \) then we have

\[
(p+kq)(7) = p(7) + (kq)(7) = p(7) + k(q(7)) = 0 + k \cdot 0 = 0 + 0 = 0,
\]

so \( p + kq \in S_1 \). Thus, \( S_1 \) contains the zero vector and is closed under addition and scalar multiplication, so is a subspace.

(b) Observe that the constant polynomial \( p(t) = 7 \) is in \( S_2 \), but \( 2p(t) = 14 \) is not. Thus, \( S_2 \) is not a subspace.
(c) The set $S_3$ is a subspace. Indeed, since $0(-t) = 0 = -0 = -0(t)$
we see that $0 \in S_3$. Moreover, we have
\[
(p + kq)(-t) = p(-t) + (kq)(-t) = p(-t) + k(q(t)) = -p(t) - kq(t)
\]
for any $p, q \in S_3$ and any $k \in \mathbb{R}$. It follows at once that $S_3$ is a
subspace.

3. The following matrix is over the reals. Find a basis for its row space,
its column space, and its null space.
\[
\begin{pmatrix}
1 & -3 & 4 & 0 & 3 \\
3 & -9 & 11 & 6 & 8 \\
-2 & 6 & -7 & -6 & -5
\end{pmatrix}
\]

**Solution:** Row reducing, we find
\[
\begin{pmatrix}
1 & -3 & 4 & 0 & 3 \\
3 & -9 & 11 & 6 & 8 \\
-2 & 6 & -7 & -6 & -5
\end{pmatrix} \sim \begin{pmatrix}
1 & -3 & 0 & 24 & -1 \\
0 & 0 & 1 & -6 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

It follows immediately that a basis for the row and column spaces are
\[
\{\begin{bmatrix} 1 & -3 & 0 & 24 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -6 & 1 \end{bmatrix}\}
\]
and
\[
\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}\right\}
\]
respectively. Using the reduced echelon form found above, we see that
the null space is determined by the system of equations
\[
x_1 - 3x_2 + 24x_4 - x_5 = 0 \quad x_3 - 6x_4 + x_5 = 0.
\]
Clearly, $x_2, x_4, x_5$ are free variables, so the parametric form of a general
solution to this system is
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -24 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
so a basis of the nullspace is

\[
\begin{bmatrix}
1 \\
0 \\
-1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
-24 \\
6 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

4. Let \( W_1 \) and \( W_2 \) be the subspaces of \( \mathbb{R}^4 \) defined by

\[
W_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad W_2 = \text{span} \left\{ \begin{pmatrix} 4 \\ 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 3 \\ 2 \end{pmatrix} \right\}.
\]

(a) Find a basis for \( W_1 + W_2 \).
(b) Find a basis for \( W_1 \cap W_2 \).
(c) Compute \( \dim(W_1 + W_2) \) and \( \dim(W_1 \cap W_2) \).

**Solution:** We row reduce the block matrix whose columns are the spanning vectors in \( W_1, W_2 \):

\[
\begin{pmatrix}
1 & 2 & 4 & 4 \\
0 & 1 & 2 & 1 \\
1 & 3 & 8 & 3 \\
0 & 2 & 4 & 2
\end{pmatrix} \sim \begin{pmatrix}
1 & 2 & 4 & 4 \\
0 & 1 & 2 & 1 \\
1 & 3 & 8 & 3 \\
0 & 2 & 4 & 2
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 2 & 1 \\
0 & 0 & 2 & -2 \\
0 & 0 & 0 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

By looking at the columns in which the leading ones occur, we deduce that a basis for \( W_1 + W_2 \) is

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 8 \\ 4 \end{bmatrix} \right\}.
\]

It follows that \( \dim(W_1 + W_2) = 3 \). Since \( \dim(W_1) = \dim(W_2) = 2 \) by inspection, we conclude from the relation

\[
\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)
\]
that the intersection $W_1 \cap W_2$ has dimension 1. Therefore, any nonzero vector in $W_1 \cap W_2$ will form a basis. To find a nonzero vector in the intersection we seek a nonzero $w$ of the form

$$w = aC_1 + bC_2 = dC_3 + eC_4,$$

where $C_i$ is the $i$th column of the matrix above. Since row reducing does not change the relationship between the columns, we easily see (by looking at the row-reduced matrix) that

$$2C_1 + 3C_2 - C_3 = C_4,$$

and hence that the vector

$$C_3 + C_4 = \begin{bmatrix} 8 \\ 3 \\ 11 \\ 6 \end{bmatrix}$$

is in $W_1 \cap W_2$; since it is visibly nonzero, a basis of $W_1 \cap W_2$ is the set

$$\left\{ \begin{bmatrix} 8 \\ 3 \\ 11 \\ 6 \end{bmatrix} \right\}.$$

5. Suppose that $A$ is a fixed $n \times n$ matrix, $V = M_n(F)$ is the vector space of $n \times n$ matrices over the field $F$, and $T : V \longrightarrow V$ is the following function.

$$T(X) = AX -XA$$

for each $X \in V$.

(a) Show that $T$ is a linear operator on $V$ (i.e., that $T : V \rightarrow V$ is a linear mapping).

(b) Now suppose that $n = 2$ and that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let $B$ be the standard ordered basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

of $V$. Find $[T]_B$. 

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(c) Find a basis for each of \( \ker(T) \) and \( \im(T) \) (using the matrix \( A \) from part (b)).

**Solution:**

(a) Let \( X, Y \) be arbitrary \( n \times n \) matrices and \( k \) any scalar. Then

\[
T(X + kY) = A(X + kY) - (X + kY)A \\
= AX + AkY -XA -kYA \\
= (AX -XA) + k(AY -YA) \\
= T(X) + kT(Y),
\]

where we have used repeatedly the usual rules of matrix and scalar multiplication. It follows at once that \( T \) is linear.

(b) To determine the matrix of \( T \) with respect to \( B \), we must evaluate \( T \) on each basis vector and write the result as a linear combination of the basis vectors. We find:

\[
T(e_1) = \left( \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) - \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \\
= \left( \begin{array}{c} 1 \\ 2 \end{array} \right) - \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
= \left( \begin{array}{c} 0 \\ -2 \end{array} \right) = \begin{pmatrix} -2 \\ 2 \end{pmatrix} e_2 + \begin{pmatrix} 2 \\ -2 \end{pmatrix} e_3.
\]

Similarly, we compute:

\[
T(e_2) = -2e_1 + 2e_4 \quad T(e_3) = 2e_1 - 2e_4 \quad T(e_4) = 2e_2 - 2e_3.
\]

Therefore, we have

\[
[T]_B = \begin{bmatrix}
0 & -2 & 2 & 0 \\
-2 & 0 & 0 & 2 \\
2 & 0 & 0 & -2 \\
0 & 2 & -2 & 0
\end{bmatrix}.
\]

(c) To find a basis of the kernel and image of \( T \), we first find a basis of the nullspace and column space of \( [T]_B \). To do this, we row
reduce $[T]_B$:

\[
\begin{bmatrix}
0 & -2 & 2 & 0 \\
-2 & 0 & 0 & 2 \\
2 & 0 & 0 & -2 \\
0 & 2 & -2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Accordingly, a basis of the column span of $[T]_B$ is

\[
\begin{bmatrix}
0 \\
-2 \\
2 \\
0
\end{bmatrix},
\begin{bmatrix}
-2 \\
0 \\
0 \\
2
\end{bmatrix}.
\]

Moreover, from the row reduction of $[T]_B$ above, we find that

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix}
\]

is a basis of the nullspace of $[T]_B$. We conclude that

\[
\begin{bmatrix}
0 & -2 \\
2 & 0
\end{bmatrix},
\begin{bmatrix}
-2 & 0 \\
0 & 2
\end{bmatrix}
\]

is a basis of $\text{im}(T)$ and

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

is a basis of $\ker(T)$.

6. Suppose that $A$ is an $n \times n$ matrix over a field $F$. Show that the following conditions on the matrix $A$ are equivalent.

(a) $A$ is invertible.

(b) For every $n \times n$ matrix $B$ over $F$, there is a solution to the matrix equation $AX = B$.

(c) For every $n \times n$ matrix $B$ over $F$, there is a unique solution to the matrix equation $AX = B$. 

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Solution: First observe that if $AX = B$ has a unique solution, then in particular it has a solution, so $c) \implies b)$. Assuming $b)$ and taking $B = I_n$, we conclude that $AX = I$ has a solution; this implies that $A$ is invertible so $b) \implies a)$. Now assume that $a)$ holds. Then if $B$ is any $n \times n$ matrix, $X = A^{-1}B$ is a solution to $AX = B$. Given another solution $X'$, we have $AX = B = AX'$ so $A^{-1}AX = A^{-1}AX'$ whence $X = X'$, and it follows that $X = A^{-1}B$ is the unique solution to $AX = B$. Thus $a) \implies c)$. We conclude that all three statements are equivalent.