MATH 223, Linear Algebra
Fall, 2007
Assignment 5, due *in class* October 15, 2007

1. $W_1 = span\left\{\begin{pmatrix}1\\0\\5\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\15\\4\end{pmatrix}, \begin{pmatrix}1\\-1\\5\\-12\end{pmatrix}\right\}$ and $W_2 = span\left\{\begin{pmatrix}5\\0\\25\\-9\end{pmatrix}, \begin{pmatrix}13\\2\\65\\-5\end{pmatrix}, \begin{pmatrix}-11\\-4\\-55\\-17\end{pmatrix}\right\}$ are subspaces of $\mathcal{R}^4$. Find a basis for each of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

2. Let $V = M_3(\mathcal{R})$ be the real vector space of 3×3 matrices with real entries. Let $A = \begin{pmatrix}3 & 5 & 2\\1 & 0 & -1\\7 & 5 & -2\end{pmatrix}$. Now let $T : V \longrightarrow V$ be defined by $T(X) = AXA^T$ for any $X \in V$.

   (a) Show that T is a linear operator on V.

   (b) Suppose that

   $$B = \left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\right\},$$

   $$\left\{\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right\}$$

   is the standard ordered basis for V. Find $[T]_B$.

   (c) Find a basis for each of $ker(T)$ and $im(T)$.

3. Show the *exchange property* for linear span. That is, suppose that V is a vector space over the field F and $S \cup \{\vec{v}, \vec{w}\}$ is a subset of F; also suppose that $\vec{w} \in span(S \cup \{\vec{v}\})$ but $\vec{w} \notin span(S)$. Show that (in this case) $\vec{v} \in span(S \cup \{\vec{w}\})$.

4. Find the inverse of the following matrix, and express it as a product of elementary matrices. (It is, of course, over $\mathcal{C}$, the complex numbers.) $\begin{pmatrix}i & 0 & 2-i\\0 & 1 & 0\\1+3i & 0 & 5-3i\end{pmatrix}$.

5. Let $V = P_5(t)$ be the real vector space of polynomials of degree at most 5. Which of the following subsets of V are subspaces of it? Justify your answers.

(a) $S_1 = \{p \in V : \frac{1}{4}(p'(t))^2 = p(t)\}$.

(b) $S_2 = \{p \in V : p(2) = p(-2) = 0\}$.

(c) $S_3 = \{p \in V : tp'(t) = 5p(t)\}$.

6. Find a basis for each of the row space, column space, and null space of the following matrix. It is over $\mathcal{Z}_2$. $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$. What is the dimension of each of these spaces, and how many elements does each have?