MATH 223, Linear Algebra
Fall, 2007
Assignment 6, due *in class* Friday November 2, 2007

1. Diagonalize the following matrices over $\mathcal{R}$:
   (a) $\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$.
   (a) Explain why A can not be diagonalized over $\mathcal{R}$.
   (b) Diagonalize A over $\mathcal{C}$.

3. Let V be a finite dimensional vector space over a field K, and suppose that $T : V \to V$ is a linear operator on V with eigenvalues $\lambda_1, \ldots, \lambda_m \in K$. For any nonnegative integer j, show that the linear operator $T^j : V \to V$ has eigenvalues $\lambda_1^j, \ldots, \lambda_m^j$.

4. Let $V = P_2(\mathcal{C})$ be the complex vector space of polynomials with coefficients in $\mathcal{C}$ of degree at most 2, and consider the linear operator
$$L : V \to V$$
defined by
$$L(a_0 + a_1 t + a_2 t^2) = (a_0 - a_1) + (a_1 - a_2)t + (a_2 - a_0)t^2.$$
   (a) Find all the eigenvalues of L (note that V is a complex vector space!).
   (b) For each eigenvalue found in part a), determine a basis of eigenvectors for the corresponding eigenspace.
   (c) Fnd a basis B such that $[L]_B$ is a diagonal matrix and determine $[L]_B$.

5. Consider the *Fibonacci sequence* defined recursively by $F_0 = 0, F_1 = 1$ and
$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$
   (a) Compute F_n for $n \leq 10$.
   (b) Show that the F_n satisfy the matrix equation
$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$
   (c) Diagonalize the matrix
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$
   (d) Show that for all n, we have the formula
$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$