MATH 223, Linear Algebra
Fall, 2007
Assignment 7, due *in class* November 9, 2007

1. A linear operator T on a vector space V is called a *projection* if $T^2 = T$. (We will be looking at *orthogonal projections* later.)

   (a) If $T = T_A$ is represented by the matrix $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$, show that T is a projection.

   (b) Show that, if T is a projection on V, then so is $I - T$; here I is the identity operator on V.

   (c) Show that, if T is a projection on V, then $V = ker(T) \oplus Im(T)$. (Recall that $\oplus$ is the direct sum, so that not only do we have $V = ker(T) + Im(T)$ but also, $ker(T) \cap Im(T) = \{\vec{0}\}$.)

   (d) Show that, if T is a projection on V, its only possible eigenvalues are 0 and 1.

   (e) Show that, if T is a projection on V and V is finite-dimensional, then T is diagonalizable.

2. For each of the following three matrices over the reals, find its characteristic polynomial and its minimal polynomial. Decide which ones are diagonalizable. $A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 5 & -1 & 2 \\ 1 & 4 & 1 \\ 0 & 3 & 3 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 3 & 0 \\ -2 & 1 & -2 \\ -1 & 2 & -1 \end{pmatrix}$.

3. (a) Suppose that the complex matrix A is diagonalizable and has only one complex (possibly real) eigenvalue. Show that A is already diagonal.

   (b) Give a 3×3 example of a real matrix B such that B has only one real eigenvalue, that B is diagonalizable (over $\mathcal{C}$), but B is not diagonal.

4. Find the determinant of the following matrix over the reals: $\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 8 & 18 & 9 \\ 3 & 17 & 53 & 4 \\ -4 & -24 & -75 & 10 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ and $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g' & h' & j' \end{pmatrix}$ be matrices with complex entries, $det(A) = 2-i$ and $det(B) = 1+3i$. If $C = \begin{pmatrix} -3ie & b+(1+i)e & 3h+2ih' \\ -3id & a+(1+i)d & 3g+2ig' \\ -3if & c+(1+i)f & 3j+2ij' \end{pmatrix}$,

what is $det(C)$? Justify.