1. (a) Let \( V = \mathbb{R}^4 \) with its usual inner product \( \langle v, w \rangle = v \cdot w \). Let \( u = (1, 1, 1, 1) \), \( v = (2, 3, -1, 2) \), and \( w = (3, 4, 0, 3) \). Determine each of \( \| u \| \), \( \| v \| \), \( \| w \| \), \( \langle u, v \rangle \), \( \langle u, w \rangle \), \( \langle v, w \rangle \).

(b) Let \( V = \mathbb{C}^n \). Show that \( \langle v, w \rangle = v \cdot w \) is not an inner product on \( V \).

(c) Now let \( V = \mathbb{C}^3 \), equipped with its usual inner product \( \langle v, w \rangle = v \cdot \overline{w} \). Set \( u = (1 + i, 2, -3 - i) \) and \( v = (i, 3i, 5 - 2i) \). Find \( \| u \| \), \( \| v \| \), \( \langle u, v \rangle \).

2. Let \( V \) be any real inner product space, with inner product \( \langle \cdot, \cdot \rangle \). Prove that for all \( u, v \in V \)
\[
\langle u + v, u - v \rangle = \| u \|^2 - \| v \|^2.
\]

3. Let \( V = M_n(\mathbb{R}) \) be the real vector space of \( n \times n \) real matrices, let \( T \) be the subspace of \( V \) consisting of upper triangular matrices, and let \( W \) be the subspace of \( T \) consisting of diagonal matrices. For any two matrices \( A, B \in V \), let
\[
\langle A, B \rangle = \text{tr}(AB).
\]
Show that \( \langle \cdot, \cdot \rangle \) is not an inner product on \( V \), but that its restriction to \( W \) is an inner product on \( W \). Justify your answer.

4. Let \( V = M_2(\mathbb{C}) \) be the complex vector space of \( 2 \times 2 \) complex matrices, equipped with the inner product
\[
\langle A, B \rangle = \text{tr}(B^T A).
\]
Let \( W \subseteq V \) be the subspace of diagonal matrices. Find a basis for \( W^\perp \).

5. Let \( V = P_3(t) \) be the real vector space of polynomials of degree at most 3 with real coefficients. For all \( f, g \in V \) let
\[
\langle f, g \rangle := \int_{-1}^{1} f(t) g(t) \, dt.
\]

(a) Show that this defines an inner product on \( V \).

(b) Let \( p_1(t) = 1, p_2(t) = t, p_3(t) = 3t^2 - 1, p_4(t) = 5t^3 - 3t \). Show that \( B = \{p_1, p_2, p_3, p_4\} \) is an orthogonal basis of \( V \). Is \( B \) an orthonormal basis of \( V \)?

6. Let \( V \) be the real vector space of continuous real-valued functions on the interval \([1, 2]\), and for any \( f, g \in V \) let
\[
\langle f, g \rangle = \int_{1}^{2} tf(t) g(t) \, dt.
\]
Show that this defines an inner product on \( V \), and that for any \( f \in V \) we have
\[
\left( \int_{1}^{2} t^2 f(t) \, dt \right)^2 \leq \frac{15}{4} \left( \int_{1}^{2} f(t)^2 \, dt \right).
\]