1. Let \( W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 3 \\ 8 \end{pmatrix} \right\} \) be a subspace of \( \mathbb{R}^4 \). Find an orthonormal basis for each of \( W \) and \( W^\perp \). Also find the orthogonal projections \( \text{Proj}_W \vec{v} \) and \( \text{Proj}_{W^\perp} \vec{v} \), where \( \vec{v} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} \).

2. Let \( W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 + i \\ 2i \end{pmatrix}, \begin{pmatrix} 1 - i \\ 2i \end{pmatrix} \right\} \) be a subspace of \( \mathbb{C}^5 \). Find an orthonormal basis for each of \( W \) and \( W^\perp \). Also find the orthogonal projections \( \text{Proj}_W \vec{v} \) and \( \text{Proj}_{W^\perp} \vec{v} \), where \( \vec{v} = \begin{pmatrix} 2 - i \\ 2 + 2i \\ -2 + i \\ 2 - 2i \\ -1 \end{pmatrix} \).

3. For each of the following Hermitian matrices \( H_j \), find a unitary matrix \( U_j \) such that \( U_j^H H_j U \) is diagonal. Also find the diagonal matrix.

\[
H_1 = \begin{pmatrix} 2 & 1 + i \\ 1 - i & 3 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 4 & 2 + 2i & 1 - i \\ 2 - 2i & 6 & -2i \\ 1 + i & 2i & 3 \end{pmatrix}.
\]

4. (a) Suppose that \( A \) is a symmetric matrix over the reals with nonnegative eigenvalues. Show that there is a symmetric real matrix \( B \) such that \( B^2 = A \). [A matrix \( B \) such that \( B^2 = A \) is called a square root of \( A \).]

(b) Find a symmetric square root of the matrix \( \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \).

5. Suppose that \((a_1, b_1), \ldots, (a_n, b_n)\) are points in the plane \( \mathbb{R}^2 \). If \( n \geq 3 \), it’s unlikely that there is a line going through all of them.

(a) Show that the line defined by \( y = \alpha x + \beta \) goes through all of them if
and only if \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \) is a solution to \( A\vec{x} = \vec{b} \), where
\[
A = \begin{pmatrix}
   a_1 & 1 \\
   a_2 & 1 \\
   \vdots & \vdots \\
   a_n & 1
\end{pmatrix}
\]
and \( \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \).

As mentioned, this is usually impossible if \( n \geq 3 \). But the system
\( A^T A\vec{x} = A^T \vec{b} \) frequently has a unique solution.

(b) Show that, if \( A \) is as above, and not all the values \( a_1, \ldots, a_n \) are the same, then \( A^T A \) has positive determinant and hence there is a unique solution to \( A^T A\vec{x} = A^T \vec{b} \).

The solution \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \) is known as the least squares solution to the original system \( A\vec{x} = \vec{b} \), and the line defined by \( y = \alpha x + \beta \) is the line of best fit to the data \((a_1, b_1), \ldots, (a_n, b_n)\).

(c) If \( n = 4 \) and \((a_1, b_1) = (0, 2), (a_2, b_2) = (1, 3), (a_3, b_3) = (2, 5)\) and \((a_4, b_4) = (3, 6)\) find the least-squares solution to \( A\vec{x} = \vec{b} \); also find the line of best fit to these data.