MATH 223, Linear Algebra
Fall, 2007
Solutions to Assignment 1

1. Let $z = 2 - 7i$ and $w = 3 + 4i$. Find $\bar{z}$, $\bar{w}$, $z + w$, $z - w$, $z \cdot w$ and $\frac{z}{w}$ (all in the form $a + bi$ with a and b real numbers). Find the absolute value of each of these 6 numbers.

   Solution: $\bar{z} = 2 + 7i$, $\bar{w} = 3 - 4i$, $z + w = 5 - 3i$, $z - w = -1 - 11i$, $z \cdot w = (2-7i)\cdot(3+4i) = 2\cdot3+2\cdot4i-7i\cdot3-7i\cdot4i = 6+8i-21i+28 = 34-13i$ and

   $$\frac{z}{w} = \frac{2-7i}{3+4i} = \frac{(2-7i)(3-4i)}{(3+4i)(3-4i)} = \frac{6-8i-21i-28}{3^2+4^2} = \frac{-22}{25} + \frac{-29}{25}i.$$

   The absolute vales are, in order, $\sqrt{2^2+7^2} = \sqrt{53}$, $\sqrt{3^2+(-4)^2} = \sqrt{25} = 5$, $\sqrt{5^2+(-3)^2} = \sqrt{34}$, $\sqrt{(34)^2+(-13)^2} = \sqrt{1325} = 5\sqrt{53}$ and

   $$\sqrt{(\frac{-22}{25})^2 + (\frac{-29}{25})^2} = \frac{1}{25}\sqrt{22^2+29^2} = \frac{\sqrt{1325}}{25} = \frac{\sqrt{53}}{5}.$$

   (Full marks for the second-last answer.)

2. Show that if z and w are any two complex numbers, then $\overline{z \cdot w} = \bar{z}\cdot\bar{w}$. Use this to show that if A and B are any complex matrices, then $\overline{A \cdot B} = \bar{A}\cdot\bar{B}$. [N.B. The conjugate $\bar{A}$ of a matrix A is the most obvious thing — you just replace each entry of A by its conjugate. Also, we of course assume here that $A \cdot B$ is defined.]

   Solution: Suppose that $z = a + bi$ and $w = c + di$, where a, b, c, d are real. Then $z \cdot w = (ac - bd) + (ad + bc)i$, so $\overline{z \cdot w} = (ac - bd) - (ad + bc)i$. $\bar{z} = a - bi$ and $\bar{w} = c - di$, so $\bar{z}\cdot\bar{w} = ac - (-b)(-d) + (a(-d) + (-b)c)i = ac - bd - (ad + bc)i$; that does the first part.

   Obviously, all of the matrices $A\cdot B$, $\overline{A \cdot B}$ and $\bar{A}\cdot\bar{B}$ have the same number of rows and columns. We must check that the (j,k)-entry of $\overline{A \cdot B}$ is the same as the (j,k)-entry of $\bar{A}\cdot\bar{B}$ for every j and k. Suppose that A has n columns and therefore B has n rows. The (j,k)-entry of $A \cdot B$ is then $\sum_{m=1}^{n} a_{j,m}b_{m,k}$ where of course $a_{j,m}$ is the (j,m)-entry of A and $b_{m,k}$ is the (m,k)-entry of B. The (j,k)-entry of $\overline{A \cdot B}$ is the conjugate of $\sum_{j=1}^{n} a_{j,m}b_{m,k}$.

   We haven't shown that the conjugate of a sum is the sum of the conjugates, but this is very easy. (Try it yourself, or ask. Don't just take my word for it.) So the entry we're interested in is $\sum_{m=1}^{n} \overline{a_{j,m}b_{m,k}}$ and by the first part of the problem, this is $\sum_{m=1}^{n} \bar{a}_{j,m}\bar{b}_{m,k}$. This is indeed the (j,k)-entry of $\bar{A}\cdot\bar{B}$.

3. Solve each of the following systems of equations. That is, find the unique solution if there is one, the general solution in vector parametric form if there is more than one solution, or explain why there is no solution if that is the case. Use augmented matrices.

(a) This one's over the field $\mathcal{R}$, the reals. $\begin{array}{rrrrrcr} x_1 & -3x_2 & & +2x_4 & +5x_5 & = & 7 \\ 3x_1 & -6x_2 & +2x_3 & +x_4 & -2x_5 & = & 1 \\ 5x_1 & -12x_2 & +2x_3 & +5x_4 & +8x_5 & = & 15 \end{array}$

Solution: We start by row-reducing the augmented matrix $\left(\begin{array}{rrrrrcr} 1 & -3 & 0 & 2 & 5 & | & 7 \\ 3 & -6 & 2 & 1 & -2 & | & 1 \\ 5 & -12 & 2 & 5 & 8 & | & 15 \end{array}\right)$. The elementary row operations $R_2 \mapsto R_2 - 3R_1$ and $R_3 \mapsto R_3 - 5R_1$ turn it into $\left(\begin{array}{rrrrrcr} 1 & -3 & 0 & 2 & 5 & | & 7 \\ 0 & 3 & 2 & -5 & -17 & | & -20 \\ 0 & 3 & 2 & -5 & -17 & | & -20 \end{array}\right)$. Before we start introducing (horrors!) fractions, let's perform $R_1 \mapsto R_1 + R_2$ and $R_3 \mapsto R_3 - R_2$. This coughs up $\left(\begin{array}{rrrrrcr} 1 & 0 & 2 & -3 & -12 & | & -13 \\ 0 & 3 & 2 & -5 & -17 & | & -20 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{array}\right)$. Finally, $R_2 \mapsto \frac{1}{3}R_2$ doles out the RREF matrix $\left(\begin{array}{rrrrrcr} 1 & 0 & 2 & -3 & -12 & | & -13 \\ 0 & 1 & \frac{2}{3} & -\frac{5}{3} & -\frac{17}{3} & | & -\frac{20}{3} \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{array}\right)$. In a solution, x_3 can be any real r, x_4 any real s, and x_5 and real t. Then $x_1 = -13 - 2r + 3s + 12t$ and $x_2 = -\frac{20}{3} - \frac{2}{3}r + \frac{5}{3}s + \frac{17}{3}t$. In vector parametric form, the general solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -13 \\ -\frac{20}{3} \\ 0 \\ 0 \\ 0 \end{pmatrix} +$
$r \begin{pmatrix} -2 \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 12 \\ \frac{17}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 12 \\ \frac{17}{3} \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

(b) This one's also over the field $\mathcal{R}$. $\begin{array}{rrrrrcr} x_1 & -3x_2 & & +2x_4 & +5x_5 & = & 4 \\ 3x_1 & -6x_2 & +2x_3 & +x_4 & -2x_5 & = & -3 \\ 5x_1 & -12x_2 & +2x_3 & +5x_4 & +8x_5 & = & 7 \end{array}$

Solution: The astute among you will have noticed that you could have done this at the same time as the previous part. But since I have a word processor, start by row-reducing the augmented matrix $\left(\begin{array}{rrrrrcr} 1 & -3 & 0 & 2 & 5 & | & 4 \\ 3 & -6 & 2 & 1 & -2 & | & -3 \\ 5 & -12 & 2 & 5 & 8 & | & 7 \end{array}\right)$. The elementary row operations

$R_2 \mapsto R_2 - 3R_1$ and $R_3 \mapsto R_3 - 5R_1$ turn it into $\begin{pmatrix} 1 & -3 & 0 & 2 & 5 & | & 4 \\ 0 & 3 & 2 & -5 & -17 & | & -15 \\ 0 & 3 & 2 & -5 & -17 & | & -13 \end{pmatrix}$.
Before we start introducing (horrors!) fractions, let's perform $R_1 \mapsto R_1 + R_2$ and $R_3 \mapsto R_3 - R_2$. This coughs up $\begin{pmatrix} 1 & 0 & 2 & -3 & -12 & | & -11 \\ 0 & 3 & 2 & -5 & -17 & | & -15 \\ 0 & 0 & 0 & 0 & 0 & | & 2 \end{pmatrix}$.
It is clear that there are no solutions, as the last line of this matrix corresponds to the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 2$.

(c) This one's over the field $\mathcal{C}$, the complex numbers. $\begin{array}{rrcr} (1+i)x_1 & +(3-i)x_2 & = & 6+i \\ (2+2i)x_1 & +(1-5i)x_2 & = & -2i \end{array}$

Solution: Start with $\begin{pmatrix} 1+i & 3-i & | & 6+i \\ 2+2i & 1-5i & | & -2i \end{pmatrix}$. After performing $R_2 \mapsto R_2 - 2R_1$, we get $\begin{pmatrix} 1+i & 3-i & | & 6+i \\ 0 & -5-3i & | & -12-4i \end{pmatrix}$. Then $R_2 \mapsto \frac{1}{-5-3i} R_2$ gives us $\begin{pmatrix} 1+i & 3-i & | & 6+i \\ 0 & 1 & | & \frac{36}{17} - \frac{8}{17}i \end{pmatrix}$. Now $R_1 \mapsto R_1 + (-3+i)R_2$ donates $\begin{pmatrix} 1+i & 0 & | & \frac{2}{17} + \frac{77}{17}i \\ 0 & 1 & | & \frac{36}{17} - \frac{8}{17}i \end{pmatrix}$. Finally, $R_1 \mapsto \frac{1}{1+i} R_1$ yields the RREF matrix $\begin{pmatrix} 1 & 0 & | & \frac{79}{34} + \frac{75}{34}i \\ 0 & 1 & | & \frac{36}{17} - \frac{8}{17}i \end{pmatrix}$. The unique solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{79}{34} + \frac{75}{34}i \\ \frac{36}{17} - \frac{8}{17}i \end{pmatrix}$.

(d) This one's over the two-element field $\mathcal{Z}_2$. $\begin{array}{ccccccc} x_1 & & +x_3 & & +x_5 & = & 1 \\ x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 1 \\ & x_2 & & +x_4 & +x_5 & = & 1 \\ x_1 & & +x_3 & & & = & 0 \end{array}$

In this case, explicitly list all the solutions.

Solution: Start with $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & | & 1 \\ 1 & 1 & 1 & 1 & 1 & | & 1 \\ 0 & 1 & 0 & 1 & 1 & | & 1 \\ 1 & 0 & 1 & 0 & 0 & | & 0 \end{pmatrix}$ and the operations $R_2 \mapsto R_2 + R_1$, $R_4 \mapsto R_4 + R_1$, bringing forth $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & 1 \end{pmatrix}$.

Next, $R_3 \mapsto R_3 + R_2$ allots us $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & | & 1 \\ 0 & 1 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & 1 \end{pmatrix}$. To find the RREF, we now just need to do $R_1 \mapsto R_1 + R_3$ and $R_4 \mapsto R_4 + R_3$;

$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$ results. We must needs have $x_5 = 1$, but $x_3 = s$ and $x_4 = t$ are parameters. Then we have $x_1 = -s = s$ and $x_2 = -t = t$ (we're in $\mathcal{Z}_2$ here). In vector parametric form, the solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Now s and t must be either 0 or 1, so we have four solutions; $s = t = 0$ gives us $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; $s = 0, t = 1$ yields $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$; $s = 1, t = 0$ proffers $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, and finally with $s = t = 1$ we locate $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

4. (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any 2×2 matrix. Show that there is a nonzero vector $\vec{v}$ with $A\vec{v} = \vec{0}$ if and only if $ad - bc = 0$.

Solution: You should note that what I am about to do works over any field. We consider row-reducing the augmented matrix $\begin{pmatrix} a & b & | & 0 \\ c & d & | & 0 \end{pmatrix}$.

We distinguish two cases — if $a \neq 0$ and if $a = 0$.

Suppose that $a \neq 0$. Then $R_1 \mapsto \frac{1}{a} R_1$ turns our matrix into $\begin{pmatrix} 1 & \frac{b}{a} & | & 0 \\ c & d & | & 0 \end{pmatrix}$.

Next $R_2 \mapsto R_2 - cR_1$ gives us $\begin{pmatrix} 1 & \frac{b}{a} & | & 0 \\ 0 & d - \frac{bc}{a} & | & 0 \end{pmatrix}$. The entry $d - \frac{bc}{a}$ is also $\frac{ad-bc}{a}$. This is zero if and only if $ad - bc = 0$, and in that case we get the nonzero solution $\vec{v} = \begin{pmatrix} -\frac{b}{a} \\ 1 \end{pmatrix}$. If $ad - bc$ is not zero, we can multiply the second row by $\frac{a}{ad-bc}$ and then replace R_1 by $R_1 - \frac{b}{a} R_2$ and get $\begin{pmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{pmatrix}$; clearly this has only the trivial solution $x_1 = x_2 = 0$.

Now suppose that $a = 0$ and our augmented matrix is $\begin{pmatrix} 0 & b & | & 0 \\ c & d & | & 0 \end{pmatrix}$.

The only way $ad - bc = 0$ is possible is if b or c is zero. If $c = 0$, we can take $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and if $c \neq 0$ but $b = 0$, we can choose $\vec{v} = \begin{pmatrix} d \\ -c \end{pmatrix}$ — either way we get a nontrivial solution. Otherwise, b and c are both nonzero, and we can perform the row operations $R_1 \mapsto \frac{1}{b}R_1$ and $R_2 \mapsto \frac{1}{c}R_2$ to get $\begin{pmatrix} 0 & 1 & \mid & 0 \\ 1 & \frac{d}{c} & \mid & 0 \end{pmatrix}$. If we then switch rows and do $R_1 \mapsto R_1 - \frac{d}{c}R_2$, we again get $\begin{pmatrix} 1 & 0 & \mid & 0 \\ 0 & 1 & \mid & 0 \end{pmatrix}$. This shows that if $a = 0$ and $ad - bc \neq 0$, we only have the trivial solution.

[There are other ways to do this. For instance, in case $ad - bc \neq 0$, it's not hard to see that A^{-1} exists (and equals $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$). So if $ad - bc \neq 0$ but $A\vec{v} = \vec{0}$, then $A^{-1}(A\vec{v}) = A^{-1}\vec{0} = \vec{0}$, and so on. The scalar $ad - bc$ is known as the *determinant* of A, and we will look at determinants more closely later.]

(b) Find all complex numbers λ (if any) such that $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\vec{v} = \lambda\vec{v}$ has a nonzero solution $\vec{v}$.

Solution: To say $A\vec{v} = \lambda\vec{v}$ is the same thing as saying that $A\vec{v} = \lambda I\vec{v}$, which is equivalent to $(A - \lambda I)\vec{v} = \vec{0}$. To say that this has a nontrivial solution is thus to say that $A - \lambda I$ has zero determinant. For the given A, $A - \lambda I = \begin{pmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{pmatrix}$. By the first part of this problem, this has a nontrivial solution if and only if $(3-\lambda)(3-\lambda) - (-2)(2) = 0$. I'm sure that some of you then wrote $\lambda^2 - 6\lambda + 13 = 0$, but why not go straight to $(3-\lambda)^2 = -4$? From there, $3 - \lambda = \pm 2i$ and $\lambda = 3 \pm 2i$. (Incidentally, the numbers $3 + 2i$ and $3 - 2i$ are known as the *eigenvalues* of the matrix $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$. Coming attractions.)

(c) For each λ you found in the previous part, find all vectors $\vec{v}$ such that $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}\vec{v} = \lambda\vec{v}$.

Solution: We solve $(A - \lambda I)\vec{v} = \vec{0}$ for each of our two λ's, where $A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$. For $\lambda = 3 + 2i$, we consider the augmented matrix $\begin{pmatrix} -2i & -2 & \mid & 0 \\ 2 & -2i & \mid & 0 \end{pmatrix}$. This row-reduces quickly to $\begin{pmatrix} 1 & -i & \mid & 0 \\ 0 & 0 & \mid & 0 \end{pmatrix}$.

The solutions are $t\begin{pmatrix} i \\ 1 \end{pmatrix}$, one for each complex t.

For $\lambda = 3-2i$, we consider the augmented matrix $\begin{pmatrix} 2i & -2 & \mid & 0 \\ 2 & 2i & \mid & 0 \end{pmatrix}$.

This row-reduces quickly to $\left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$. The solutions are $t\begin{pmatrix} -i \\ 1 \end{pmatrix}$, one for each complex t.