MATH 223, Linear Algebra  
Fall, 2007  
Solutions to Assignment 5

1. \( W_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 15 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 5 \\ -12 \end{pmatrix} \right\} \) and \( W_2 = \text{span} \left\{ \begin{pmatrix} 5 \\ 0 \\ 25 \\ -9 \end{pmatrix}, \begin{pmatrix} 13 \\ 2 \\ 65 \\ -5 \end{pmatrix}, \begin{pmatrix} -11 \\ -4 \\ -55 \\ -17 \end{pmatrix} \right\} \) are subspaces of \( \mathbb{R}^4 \). Find a basis for each of \( W_1, W_2, W_1 + W_2 \) and \( W_1 \cap W_2 \).

Solution: We start by row-reducing the big matrix
\[
\begin{pmatrix}
1 & 3 & 1 & 5 & 13 & -11 \\
0 & 1 & -1 & 0 & 2 & -4 \\
5 & 15 & 5 & 25 & 65 & -55 \\
-2 & 4 & -12 & -9 & -5 & -17
\end{pmatrix}
\]
getting
\[
\begin{pmatrix}
1 & 0 & 4 & 0 & 2 & -4 \\
0 & 1 & -1 & 0 & 2 & -4 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]. It is evident from the left half of this that the first two columns are independent, but not all three.

A basis for \( W_1 \) is then \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 15 \\ 4 \end{pmatrix} \right\} \). Only slightly less obvious is that two of the three last columns are independent, but not all three; a basis for \( W_2 \) is \( \left\{ \begin{pmatrix} 5 \\ 0 \\ 25 \\ -9 \end{pmatrix}, \begin{pmatrix} 13 \\ 2 \\ 65 \\ -5 \end{pmatrix} \right\} \). Of the columns of the big row-reduced matrix, we can choose only three independent ones, most obviously the first, second and fourth. A basis for \( W_1 + W_2 \) is \( \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 15 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 25 \\ -9 \end{pmatrix} \right\} \). Finally, if we label the columns \( C_1, \ldots, C_6 \), then from the row-reduced matrix, it’s clear that (say) \( C_5 = 2C_1 + 2C_2 + C_4 \); i.e \( C_5 - C_4 = 2C_1 + 2C_2 \). This is true for the original big matrix and that vector \( C_5 - C_4 \) is in the intersection. Since \( 2 + 2 = 3 + 1 \), the intersection has dimension 1, and a basis for \( W_1 \cap W_2 \) is \( \left\{ \begin{pmatrix} 8 \\ 2 \\ 40 \\ 4 \end{pmatrix} \right\} \).

2. Let \( V = M_3(\mathbb{R}) \) be the real vector space of \( 3 \times 3 \) matrices with real
entries. Let \( A = \begin{pmatrix} 3 & 5 & 2 \\ 1 & 0 & -1 \\ 7 & 5 & -2 \end{pmatrix} \). Now let \( T : V \rightarrow V \) be defined by

\[ T(X) = AXA^T \text{ for any } X \in V. \]

(a) Show that \( T \) is a linear operator on \( V \).

Solution: The particular matrix \( A \) is irrelevant here. For any \( X_1, X_2 \in V \),

\[ T(X_1 + X_2) = A(X_1 + X_2)A^T = (AX_1 + AX_2)A^T = AX_1A^T + AX_2A^T = T(X_1) + T(X_2) \]

using the distributive laws.

Also, for any \( X \in V \) and scalar \( \alpha \),

\[ T(\alpha X) = A(\alpha X)A^T = \alpha AXA^T = \alpha T(X). \]

That does it.

(b) Suppose that

\[ B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \]

is the standard ordered basis for \( V \). Find \([T]_B\).

Solution: Here the particular matrix \( A \) matters. Routine matrix multiplication shows us that

\[ T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 21 \\ 3 & 1 & 7 \\ 21 & 7 & 9 \end{pmatrix}. \]

Letting the basis matrices be \( B_1, \ldots, B_9 \), we have \( T(B_1) = 9B_1 + 3B_2 + 21B_3 + 3B_4 + 1B_5 + 7B_6 + 21B_7 + 7B_8 + 49B_9 \). Those 9 coefficients go into the first column of \([T]_B\). \( T(B_2) = AB_2A^T = 15B_1 + 0B_2 + 15B_3 + 5B_4 + 0B_5 + 5B_6 + 35B_7 + 0B_8 + 35B_9 \), telling us the second column of \([T]_B\). We skip the rest of the details. \([T]_B =

\[ \begin{pmatrix} 9 & 15 & 6 & 15 & 25 & 10 & 6 & 10 & 4 \\ 3 & 0 & -3 & 3 & 0 & -5 & 2 & 0 & -2 \\ 21 & 15 & -6 & 35 & 25 & -10 & 14 & 10 & -4 \\ 3 & 5 & 2 & 0 & 0 & 0 & -3 & -5 & -2 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 7 & 5 & -2 & 0 & 0 & 0 & -7 & -5 & 2 \\ 21 & 35 & 14 & 15 & 25 & 10 & -6 & -10 & -4 \\ 7 & 0 & -7 & 3 & 0 & -5 & -2 & 0 & 2 \\ 9 & 35 & 14 & 35 & 25 & -10 & -14 & -10 & 4 \end{pmatrix} \]

(c) Find a basis for each of \( \ker(T) \) and \( \text{im}(T) \).

Solution: Sorry about this one. I promise, no \( 9 \times 9 \) matrices on the ex-
ams. When I row-reduced $[T]_B$, I got the following.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} & 0 & -\frac{7}{3} \\
0 & 1 & 0 & 0 & 0 & -\frac{10}{3} & -1 & \frac{10}{3}
\end{pmatrix}
\]

A basis for the column space of $[T]_B$ consists of its first six columns; the corresponding basis of $\text{im}(T)$ is

\[
\begin{cases}
\begin{pmatrix}
9 & 3 & 21 \\
3 & 1 & 7 \\
21 & 7 & 9 \\
25 & 0 & 25
\end{pmatrix}, \\
\begin{pmatrix}
15 & 0 & 15 \\
5 & 0 & 5 \\
35 & 0 & 35 \\
10 & -5 & -10
\end{pmatrix}, \\
\begin{pmatrix}
6 & -3 & -6 \\
2 & -1 & -2 \\
14 & -7 & 14 \\
6 & -3 & -6
\end{pmatrix}, \\
\begin{pmatrix}
15 & 3 & 35 \\
5 & 0 & 5 \\
35 & 0 & 35 \\
15 & 3 & 35
\end{pmatrix}, \\
\begin{pmatrix}
25 & 0 & 25 \\
0 & 0 & 0 \\
25 & 0 & 25 \\
10 & -5 & -10
\end{pmatrix}
\end{cases}
\]

To find the null space of $[T]_B$ we solve the homogeneous system and get

\[
x_1 = -\frac{7}{3}r + 0s + \frac{7}{11}t, \\
x_2 = \frac{10}{3}r + 1s - \frac{10}{11}t, \\
x_3 = -\frac{10}{3}r + 0s + \frac{10}{11}t, \\
x_4 = \frac{7}{3}r + 0s + \frac{10}{11}t, \\
x_5 = -\frac{43}{29}r + -1s + \frac{21}{4}t, \\
x_6 = \frac{23}{29}r + 0s + \frac{1}{4}t, \\
x_7 = 1r + 0s + 0t, \\
x_8 = 0r + 1s + 0t, \\
x_9 = 0r + 0s + 1t
\]

We get 3 matrices in our basis for $\ker(T)$; the basis is

\[
\begin{cases}
\begin{pmatrix}
-\frac{7}{3} & \frac{10}{3} & -\frac{10}{3} \\
\frac{5}{3} & -\frac{29}{3} & \frac{2}{3} \\
1 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
0 & 1 & 0 \\
0 & -1 & 0 \\
0 & 1 & 0
\end{pmatrix}, \\
\begin{pmatrix}
-\frac{7}{11} & -\frac{13}{11} & \frac{10}{11} \\
\frac{7}{11} & \frac{21}{11} & \frac{10}{11} \\
0 & 0 & 0
\end{pmatrix}
\end{cases}
\]

[Note to the markers: I hope you don’t get too many pages of hairy calculations. The main things I’m looking for are: the matrix $[T]_B$ should be 9 × 9 and they should know in principle how to calculate it — that’s not so hard; the row-reduction is a bit much, but they should at the very least know that the dimensions of the kernel and image add up to 9; also their bases should come out as sets of 3 × 3 matrices. If they get these right, don’t worry too much about the numbers being off.]

3. Show the exchange property for linear span. That is, suppose that $V$ is a vector space over the field $F$ and $S \cup \{\vec{v}, \vec{w}\}$ is a subset of $F$; also suppose that $\vec{w} \in \text{span}(S \cup \{\vec{v}\})$ but $\vec{w} \notin \text{span}(S)$. Show that (in this case) $\vec{v} \in \text{span}(S \cup \{\vec{w}\})$.
Solution: Since \( \vec{w} \in \text{span}(S \cup \{ \vec{v} \}) \), there must be some vectors \( \vec{u}_1, \ldots, \vec{u}_n \in S \) and some scalars \( a_1, \ldots, a_n \) and \( b \) such that \( \vec{w} = a_1 \vec{u}_1 + \cdots + a_n \vec{u}_n + b \vec{v} \). Since \( \vec{w} \notin \text{span}(S) \), \( b \neq 0 \). So we can solve for \( \vec{v} \). To wit, \( \vec{v} = -\frac{a_1}{b} \vec{u}_1 - \cdots - \frac{a_n}{b} \vec{u}_n + \frac{1}{b} \vec{w} \). So indeed \( \vec{v} \in \text{span}(S \cup \vec{w}) \).

4. Find the inverse of the following matrix, and express it as a product of elementary matrices. (It is, of course, over \( \mathbb{C} \), the complex numbers.)

\[
\begin{bmatrix}
i & 0 & 2 - i \\
0 & 1 & 0 \\
1 + 3i & 0 & 5 - 3i
\end{bmatrix}
\]

Solution: We work with the big matrix

\[
\begin{bmatrix}
i & 0 & 2 - i & | & 1 & 0 & 0 \\
0 & 1 & 0 & | & 0 & 1 & 0 \\
1 + 3i & 0 & 5 - 3i & | & 0 & 0 & 1
\end{bmatrix}
\]

and row-reduce it step-by-step. Our first row operation is \( R_1 \mapsto -iR_1 \), corresponding to the elementary matrix

\[
E_1 = \begin{bmatrix}
-i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and yields

\[
\begin{bmatrix}
1 & 0 & -1 - 2i & | & -i & 0 & 0 \\
0 & 1 & 0 & | & 0 & 1 & 0 \\
1 + 3i & 0 & 5 - 3i & | & 0 & 0 & 1
\end{bmatrix}
\]

Our second row operation is \( R_3 \mapsto R_3 + (-1 - 3i)R_1 \), corresponding to the elementary matrix

\[
E_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 - 3i & 0 & 1
\end{bmatrix}
\]

and yields

\[
\begin{bmatrix}
1 & 0 & -1 - 2i & | & -i & 0 & 0 \\
0 & 1 & 0 & | & 0 & 1 & 0 \\
0 & 0 & 2i & | & -3 + i & 0 & 1
\end{bmatrix}
\]

Our third row operation is \( R_3 \mapsto -\frac{1}{2}iR_3 \), corresponding to the elementary matrix

\[
E_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -\frac{1}{2}i
\end{bmatrix}
\]

and yields

\[
\begin{bmatrix}
1 & 0 & -1 - 2i & | & -i & 0 & 0 \\
0 & 1 & 0 & | & 0 & 1 & 0 \\
0 & 0 & 1 & | & \frac{1}{2} + \frac{3}{2}i & 0 & -\frac{1}{2}i
\end{bmatrix}
\]

Our fourth and final row operation is \( R_1 \mapsto R_1 + (1 + 2i)R_3 \), corresponding to the elementary matrix

\[
E_4 = \begin{bmatrix}
1 & 0 & 1 + 2i \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and yields

\[
\begin{bmatrix}
1 & 0 & 0 | -\frac{5}{2} + \frac{3}{2}i & 0 & 1 - \frac{1}{2}i \\
0 & 1 & 0 | 0 & 1 & 0 \\
0 & 0 & 1 | \frac{1}{2} + \frac{3}{2}i & 0 & -\frac{1}{2}i
\end{bmatrix}
\]

The inverse is then

\[
\begin{bmatrix}
-\frac{5}{2} + \frac{3}{2}i & 0 & 1 - \frac{1}{2}i \\
0 & 1 & 0 \\
\frac{1}{2} + \frac{3}{2}i & 0 & -\frac{1}{2}i
\end{bmatrix}
\]

It equals \( E_4E_3E_2E_1 \), so the given matrix is then

\[
E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} =
\begin{bmatrix}
i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 + 3i & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 2i & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -1 - 2i \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
5. Let $V = P_5(t)$ be the real vector space of polynomials of degree at most 5. Which of the following subsets of $V$ are subspaces of it? Justify your answers.

(a) $S_1 = \{ p \in V : \frac{1}{2}(p'(t))^2 = p(t) \}$.
(b) $S_2 = \{ p \in V : p(2) = p(-2) = 0 \}$.
(c) $S_3 = \{ p \in V : tp'(t) = 5p(t) \}$.

Solution: $S_1$ is not a subspace, because for instance $t^2 \in S_1$, but $2t^2 \notin S_1$.

$S_2$ is a subspace, because clearly the zero polynomial satisfies the condition. Also, if $p_1$ and $p_2$ are in $S_2$, then $p_1(2) = p_1(-2) = p_2(2) = p_2(-2) = 0$, so that $(p_1 + p_2)(2) = p_1(2) + p_2(2) = 0 + 0 = 0$ and also $(p_1 + p_2)(-2) = p_1(-2) + p_2(-2) = 0 + 0 = 0$. Hence $p_1 + p_2 \in S_2$ and it is closed under addition. Finally, if $p \in S_2$ and $r$ is a real number $p(2) = p(-2) = 0$ so $(rp)(2) = rp(2) = r \cdot 0 = 0$ and also $(rp)(-2) = rp(-2) = r \cdot 0 = 0$; thus $rp \in S_2$ and $S_2$ is also closed under scalar multiplication.

$S_3$ is also a subspace. Again, the zero polynomial satisfies the condition. If $p_1$ and $p_2$ are in $S_3$, then $tp'_1(t) = 5p_1(t)$ and $tp'_2(t) = 5p_2(t)$. Hence $t(p_1+p_2)'(t) = t(p'_1(t)+p'_2(t)) = tp'_1(t)+tp'_2(t) = 5p_1(t)+5p_2(t) = 5(p_1+p_2)(t)$.

So $p_1 + p_2 \in S_3$. Finally, if $p \in S_3$ and $r$ is real, then $t(rp)'(t) = tr(p'(t)) = rtp'(t) = r(5p(t)) = 5(rp)(t)$ and $rp \in S_3$.

6. Find a basis for each of the row space, column space, and null space of the following matrix. It is over $\mathbb{Z}_2$.

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
$$

What is the dimension of each of these spaces, and how many elements does each have?

Solution: The given matrix row-reduces quickly to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

A basis for the row space is then $\{(1 \ 0 \ 0 \ 0 \ 0), (0 \ 1 \ 0 \ 1 \ 1), (0 \ 0 \ 1 \ 0 \ 1)\}$.

A basis for the null space is $\{(1 \ 0 \ 0 \ 0 \ 0), (1 \ 1 \ 1 \ 1 \ 0), (1 \ 1 \ 1 \ 1 \ 0)\}$. Finally, for the null space, we see that we can choose parameters for $x_4$ and $x_5$; we
have $x_3 = x_5$, $x_2 = x_4 + x_5$ and $x_1 = 0$. A basis for the null space is
\[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

The row and column spaces each have dimension 3 and 8 elements; the null space has dimension 2 and 4 elements.