MATH 223, Linear Algebra
Fall, 2007
Assignment 8, due *in class* Friday November 16, 2007

1. (a) Let $V = \mathcal{R}^4$ with its usual inner product $\langle v, w\rangle = v \cdot w$. Let $u = (1, 1, 1, 1)$, $v = (2, 3, -1, 2)$, and $w = (3, 4, 0, 3)$. Determine each of

$$\|u\|,\ \|v\|,\ \|w\|, \langle u, v\rangle,\ \langle u, w\rangle,\ \langle v, w\rangle.$$

(b) Let $V = \mathcal{C}^n$. Show that $\langle v, w\rangle = v \cdot w$ is not an inner product on V.

(c) Now let $V = \mathcal{C}^3$, equipped with its usual inner product $\langle v, w\rangle = v \cdot \overline{w}$. Set $u = (1 + i, 2, -3 - i)$ and $v = (i, 3i, 5 - 2i)$. Find

$$\|u\|,\ \|v\|,\ \langle u, v\rangle.$$

Solution:

(a) We have

$$\|u\| = \sqrt{u \cdot u} = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2.$$

Similarly, $\|v\| = \sqrt{18}$, and $\|w\| = \sqrt{22}$. Also, we compute:

$$\langle u, v\rangle = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot (-1) + 1 \cdot 2 = 2 + 3 - 1 + 2 = 6.$$

Similarly, $\langle u, w\rangle = 10$ and $\langle v, w\rangle == 24$.

(b) There are many problems: the given function is *not* conjugate linear or conjugate-symmetric, and it is not positive definite as, for example, if $v = (1, i, 0, \dots, 0)$ then $v \neq 0$ but $v \cdot v = 1^2 + i^2 = 0$.

(c) We compute:

$$\|u\| = \sqrt{u \cdot \overline{u}} = \sqrt{(1 + i)(1 - i) + 2 \cdot 2 + (-3 - i)(-3 + i)} = \sqrt{2 + 4 + 10} = \sqrt{16} = 4.$$

Similarly, $\|v\| = \sqrt{1 + 9 + 29} = \sqrt{39}$. We also find

$$\langle u, v\rangle = u \cdot \overline{v} = (1 + i)(-i) + 2(-3i) + (-3 - i)(5 + 2i) = -12 - 18i.$$

2. Let V be any real inner product space, with inner product $\langle \cdot, \cdot\rangle$. Prove that for all $u, v \in V$

$$\langle u + v, u - v\rangle = \|u\|^2 - \|v\|^2.$$

Solution: By linearity in the first coordinate, we have

$$\langle u + v, u - v\rangle = \langle u, u - v\rangle + \langle v, u - v\rangle.$$

Expanding each term above using linearity in the second coordinate, this equals

$$\langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle = \|u\|^2 - \|v\|^2.$$

3. Let $V = M_n(\mathcal{R})$ be the real vector space of $n \times n$ real matrices, let T be the subspace of V consisting of upper triangluar matrices, and let W be the subspace of T consisting of diagonal matrices. For any two matrices $A, B \in V$, let

$$\langle A, B\rangle = \mathrm{tr}(AB).$$

Show that $\langle\cdot,\cdot\rangle$ is *not* an inner product on V, but that its restriction to W is an inner product on W. Is the restriction of $\langle\cdot,\cdot\rangle$ to T an inner product on T? Justify your answer.

Solution: Although the given function is symmetric (try to prove this!) and linear (proof?), it is *not* positive definite on V. For example, if A is the $n \times n$ matrix with zeroes everywhere except in the $(1, n)$ position, then $\langle A, A\rangle = 0$ but $A \neq 0$. Now if $A = [a_{ij}]$ is upper triangular, so $a_{ij} = 0$ for $j > i$, then we compute that the (i, i)-entry of A^2 is

$$\sum_k a_{ik}a_{ki} = a_{ii}^2,$$

since the sum has only one nonzero term (as $a_{ik} = 0$ for $k > i$ and $a_{ki} = 0$ for $k < i$). Thus, $\mathrm{tr}(A^2) = \sum_i a_{ii}^2$. This number is at least nonnegative, but it can be zero even when A is not (if A is any upper-triangular matrix with zeroes on the diagonal, then $\mathrm{tr}(A^2) = 0$ by the formula we just derived). However, is $A = [a_{ij}]$ is *diagonal*, then $\sum_i a_{ii}^2 = 0$ if and only if $a_{ii} = 0$ for all i, and this occurs if and only if $A = 0$ (since A is diagonal). We conclude that the given (bilinear and symmetric) function *is* an inner product on W but *not* on V nor on T.

4. Let $V = M_2(\mathcal{C})$ be the complex vector space of 2×2 complex matrices, equipped with the inner product

$$\langle A, B\rangle = \mathrm{tr}(\overline{B}^T A).$$

Let $W \subseteq V$ be the subspace of diagonal matrices. Find a basis for $W^\perp$.

Solution: A basis for W is given by the matrices

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_4 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

An arbitrary matrix

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$$

is in $W^\perp$ if and only if $\langle x, e_1\rangle = 0$ and $\langle x, e_4\rangle = 0$. Using the definition of the inner product, we easily find

$$\langle x, e_1\rangle = a \qquad \langle x, e_4\rangle = d.$$

It follows that $x \in W^\perp$ if and only if $a = d = 0$. Thus, $W^\perp$ is the span of

$$B := \left\{ e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Since B is obviously linearly independent, it is a basis for $W^\perp$.

5. Let $V = P_3(t)$ be the real vector space of polynomials of degree at most 3 with real coefficients. For all $f, g \in V$ let

$$\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)dt.$$

(a) Show that this defines an inner product on V.

(b) Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = 3t^2 - 1$, $p_4(t) = 5t^3 - 3t$. Show that $B = \{p_1, p_2, p_3, p_4\}$ is an orthogonal basis of V. Is B an orthonormal basis of V?

Solution:

(a) We have

$$\langle g, f\rangle = \int_{-1}^{1} g(t)f(t)dt = \int_{-1}^{1} f(t)g(t)dt = \langle f, g\rangle,$$

so $\langle \cdot, \cdot \rangle$ is symmetric. If $f, g, h \in V$ and $a \in \mathcal{R}$ then

$$\langle af + g, h\rangle = \int_{-1}^{1} (af(t) + g(t))h(t)dt = a\int_{-1}^{1} f(t)h(t)dt + \int_{-1}^{1} g(t)h(t)dt = a\langle f, h\rangle + \langle g, h\rangle$$

so $\langle \cdot, \cdot \rangle$ is linear. Finally, we have

$$\langle f, f\rangle = \int_{-1}^{1} f(t)^2 dt \geq 0$$

with equality if and only if $f(t) = 0$ for all $t \in [-1, 1]$, and hence $f(t) = 0$ for all t (because $f(t)^2 \geq 0$ for all t, with equality for all $t \in [-1, 1]$ if and only if $f(t) = 0$ for all t). Thus, $\langle \cdot, \cdot \rangle$ is positive definite. It therefore defines an inner product.

(b) Since the degrees of the given polynomials are all distinct and there are 4 of them, the set B is clearly a basis. To show that B is orthogonal, we must show that $\int_{-1}^{1} p_i(t)p_j(t)dt = 0$ for $i \neq j$. This is an elementary computation with integrals. Notice, before doing any work, that

$$\langle p_1, p_2\rangle = \langle p_1, p_4\rangle = \langle p_2, p_3\rangle = \langle p_3, p_4\rangle = 0$$

since in every case the products $p_i p_j$ that occur in the integrals are *odd* functions of t, and thae range of integration is symmetric about the origin. Thus, we need only show that $\langle p_1, p_3\rangle = \langle p_2, p_4\rangle = 0$. We compute

$$\langle p_1, p_3\rangle = \int_{-1}^{1} p_1(t)p_3(t)dt = \int_{-1}^{1} 3t^2 - 1dt = t^3 - t\Big|_{-1}^{1} = 0$$

and

$$\langle p_2, p_4\rangle = \int_{-1}^{1} p_2(t)p_4(t)dt = \int_{-1}^{1} 5t^4 - 3t^2dt = t^5 - t^3\Big|_{-1}^{1} = 0.$$

Thus, B is orthogonal.

It is easy to see that B is *not* orthonormal, however, as

$$\langle p_1, p_1\rangle = \int_{-1}^{1} dt = t\Big|_{-1}^{1} = 2 \neq 1.$$

6. Let V be the real vector space of continuous real-valued functions on the interval $[1, 2]$, and for any $f, g \in V$ let

$$\langle f, g \rangle = \int_1^2 t f(t) g(t) dt.$$

Show that this defines an inner product on V, and that for any $f \in V$ we have

$$\left(\int_1^2 t^2 f(t) dt \right)^2 \leq \frac{15}{4} \left(\int_1^2 t f(t)^2 dt \right).$$

Solution:

We have

$$\langle g, f \rangle = \int_1^2 t g(t) f(t) dt = \int_1^2 t f(t) g(t) dt = \langle f, g \rangle,$$

so $\langle \cdot, \cdot \rangle$ is symmetric. If $f, g, h \in V$ and $a \in \mathcal{R}$ then

$$\langle af + g, h \rangle = \int_1^2 t(af(t) + g(t))h(t) dt = a \int_1^2 t f(t) h(t) dt + \int_1^2 t g(t) h(t) dt = a\langle f, h \rangle + \langle g, h \rangle$$

so $\langle \cdot, \cdot \rangle$ is linear. Finally, we have

$$\langle f, f \rangle = \int_1^2 t f(t)^2 dt \geq 0$$

with equality if and only if $f(t) = 0$ for all $t \in [1, 2]$ (because $t f(t)^2 \geq 0$ for all $t \in [1, 2]$, with equality for all $t \in [1, 2]$ if and only if $f(t) = 0$ for all such t). Thus, $\langle \cdot, \cdot \rangle$ is positive definite. It therefore defines an inner product on V.

Applying the Cauchy-Schwartz inequality

$$\langle f, g \rangle^2 \leq \langle f, f \rangle \cdot \langle g, g \rangle$$

with $g(t) = t \in V$ and $f(t) \in V$ arbitrary, we find

$$\left(\int_1^2 t^2 f(t) dt \right)^2 \leq \left(\int_1^2 t f(t)^2 dt \right) \cdot \left(\int_1^2 t^3 dt \right).$$

We easily compute

$$\int_1^2 t^3 dt = \frac{t^4}{4} \Big|_1^2 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4};$$

suhstituting this in above gives the desired inequality.