Honors Algebra 4, MATH 371 Winter 2010

Assignment 6

Due Wednesday, March 24 at 08:35

1. Let K/F be a degree 2 extension of fields.
   (a) If the characteristic of F is not 2, prove that $K = F(a)$ for some $a \in K \setminus F$ with $a^2 \in F$.
   (b) Give a counterexample to (1a) if F has characteristic 2.
   (c) Fix F of characteristic not 2 and let K_1, K_2 be quadratic extensions of F with $K_1 = F(a_1)$ and $K_2 = F(a_2)$ where $a_i^2 = b_i \in F$. Prove that $K_1 \simeq K_2$ as extensions of F (i.e. that there exists an isomorphism of fields $K_1 \simeq K_2$ restricting to the identity on F) if and only if $b_1/b_2 \in (F^\times)^2$ is a square. Conclude that the isomorphism classes of quadratic extensions of F are in bijection with the group $F^\times/(F^\times)^2$.
   (d) Using (1c), give a complete list (without repetition) of all isomorphism classes of quadratic extensions of $\mathbf{Q}$.

2. For $a \in \mathbf{F}_p$, set
$$f_a(x) := X^p - X - a \in \mathbf{F}_p[X].$$
   (a) If $a = 0$, show that $f_a(X) = \prod_{u \in \mathbf{F}_p}(X - u)$.
   (b) Suppose that $a \neq 0$ and let E_a be a splitting field of $f_a(X)$. If $r_1, r_2 \in E_a$ are roots of f_a, prove that $r_1 - r_2 \in \mathbf{F}_p$.
   (c) Show that $f_a(X)$ is irreducible for all $a \in \mathbf{F}_p^\times$.
   (d) Prove that $f_b(X)$ splits completely over E_a for each fixed $a \in \mathbf{F}_p^\times$ and all $b \in \mathbf{F}_p^\times$. Conclude that E_a is independent of a.

3. Find the minimal polynomials of $2\cos(2\pi/5)$ and $2\cos(2\pi/7)$ over $\mathbf{Q}$.

4. For each of the following extensions, determine $[K : F]$ and an F-basis of K.
   (a) $F = \mathbf{Q}$, $L = \mathbf{Q}(a, b)$ with $a^2 = 6$ and $b^3 = 2$.
   (b) $F = \mathbf{C}(T)$ and L is the splitting field of $X^n - T$ over F, with n a positive integer.
   (c) $F = \mathbf{F}_p(T)$ and L is the splitting field of $X^p - T$ over F, with p a prime.

5. Let K/F be a finite extension of fields and let $\alpha \in K$. Then α induces an F-linear map of finite-dimensional F-vector spaces
$$m_\alpha : K \to K.$$
   (a) Prove that α is a root of the characteristic polynomial of the linear map m_α. Hint: select a suitable F-basis of $F(\alpha)$.
   (b) Use (5a) to find a monic degree 3 polynomial with $\mathbf{Q}$-coefficients satisfied by $1 + \sqrt[3]{2} + \sqrt[3]{4}$.
   (c) Prove that if $K = F(\alpha)$, then the characteristic polynomial of m_α as a linear map $K \to K$ is in fact the minimal polynomial of α over F.

6. For each of the following algebraic elements α of the given field extension $K/\mathbf{Q}$, express $1/\alpha$ and $1/(\alpha + 1)$ as polynomials in α with $\mathbf{Q}$-coefficients.

(a) K is the splitting field of $f = X^3 - 3X + 1$ and α is a root of f.

(b) K is the splitting field of $f = X^4 + X^3 + x^2 + x + 1$ and α is a root of f.

(c) K is the splitting field of $f = X^5 - 3X + 3$ and α is a root of f.

7. Prove that $X^4 - 5$ is irreducible over $\mathbf{Q}$ and has splitting field K of degree 8 over $\mathbf{Q}$. Describe this splitting field explicitly as $\mathbf{Q}(a, b)$ where a is a root of $X^4 - 5$ and $b^2 \in \mathbf{Q}$. In terms of a and b, write down a $\mathbf{Q}$-basis for K.

8. Describe the splitting fields of $f := X^3 - 5$ over $\mathbf{F}_{11}$ and $\mathbf{F}_7$ and factor f into linear factors over each extension.