MATH 593
Assignment # 1
(Due Monday, September 16)

HAND IN: #'s 1,2, 3,4, 5(a,c)

NOTE: 5(b) is optional.

(#1). Let R be a commutative ring with 1, and consider the polynomial ring $R[X]$ in one variable over R. Show that $R[X]$ is an integral domain if and only if R is.

(#2). Let $R = \mathbf{C}[[x]]$ denote the ring of power series with complex coefficients.

(a). Prove that the group of units of R consists precisely of those power series $\sum a_n x^n$ such that $a_0 \neq 0$.

(b). Find the inverse of $1 - x$ in $\mathbf{C}[[x]]$.

(The same statements are valid for formal power series with coefficients in any field.)

(#3). Let $R = M_2(k)$ be the ring of 2×2 matrices over a field k. Show that R has no non-trivial two-sided ideals.

(#4). Let G be a finite group, and let $R = \mathbf{C}[G]$ be the complex group ring of G. Suppose given a homomorphism $\chi : G \longrightarrow \mathbf{C}^*$, and consider the element

$$t_\chi = \sum_{g \in G} \chi(g) \cdot [g] \ \in \ \mathbf{C}[G].$$

Show that

$$I_\chi =_{\text{def}} \{\lambda \cdot t_\chi \mid \lambda \in \mathbf{C}\}$$

is a two-sided ideal of R. Conversely, prove that any (two-sided) ideal $I \subseteq \mathbf{C}[G]$ which is a one dimensional complex vector space is of the form I_χ for some $\chi : G \longrightarrow \mathbf{C}^*$.

[CHALLENGE PROBLEM (not to hand in): Starting with a homomorphism

$$\rho : G \longrightarrow GL_2(\mathbf{C})$$

(or more generally $\rho : G \longrightarrow GL_n(\mathbf{C})$), construct some other (two-sided) ideals in $\mathbf{C}[G]$.]

(# 5). Let $\mathcal{C}(\mathbf{R})$ (resp. $\mathcal{C}([0,1])$ denote the ring of real-valued continuous functions on $\mathbf{R}$ (resp. on the closed unit interval $[0,1]$).

(a). Given $a \in \mathbf{R}$, let $I_a \in \mathcal{C}(\mathbf{R})$ be the set of all functions vanishing at a:

$$I_a \;=\; \{f \in \mathcal{C}(\mathbf{R}) \mid f(a) = 0\}.$$

Show that I_a is a maximal ideal of $\mathcal{C}(\mathbf{R})$. If $I_a \subset \mathcal{C}([0,1])$ is defined in the analogous fashion, then I_a is likewise a maximal ideal of $\mathcal{C}([0,1])$.

*(b). Let $I \subset \mathcal{C}([0,1])$ be an ideal with the property that for every $a \in [0,1]$, there exists a function $f_a \in I$ such that $f_a(a) \neq 0$. Prove that then $I = \mathcal{C}([0,1])$. Deduce that every maximal ideal of $\mathcal{C}([0,1])$ is of the form I_a for some $a \in [0,1]$.

[NOTE: This is as much an analysis as an algebra problem. You will want to use the compactness of $[0,1]$, plus the existence of partitions of unity (cf. Lang, Real Analysis, Chapt IX, §5). In the present situation, this means that given a finite open covering $\{U_i\}$ of $[0,1]$, there exist non-negative continuous functions $g_i : [0,1] \longrightarrow \mathbf{R}$, with $\operatorname{supp}(g_i) \subset U_i$, such that $\sum g_i(x) = 1$ for all $x \in [0,1]$. If you haven't had analysis, or you really object to doing analysis on an algebra problem set, then you don't need to worry about this part of the problem. I'm mainly trying illustrate in a simple setting the fact that there are many interesting connections between algebra and other fields. At the other extreme, if you enjoy this problem, you can try to obtain an analogous description of the maximal ideals in the ring $\mathcal{C}(X)$ of real-valued continuous functions on any compact Hausdorff space X.)

(c). Show that there are proper ideals $I \subsetneq \mathcal{C}(\mathbf{R})$ satisfying the analogue of the hypothesis of (b).