MATH 593
Assignment # 10
(Due Wednesday, Dec. 4)

Hand In: All three problems.

(#1). Let V be a vector space of dimension n over a field k.

(a). Prove that there is a canonical isomorphism

$$V^* \otimes \Lambda^n V \;\cong\; \Lambda^{n-1} V,$$

where as usual $V^* = \mathrm{Hom}_k(V, k)$ is the dual space of V. (By canonical, I mean independent of any choices.) Since $\Lambda^n V$ is one-dimensional, this means that V^* is (non-canonically) isomorphic to $\Lambda^{n-1}V$.

(b). Fix $k < n$, and let $\omega \in \Lambda^k V$. Show that if

$$\omega \wedge v \;=\; 0 \in \Lambda^{k+1}(V)$$

for every $v \in V$, then $\omega = 0 \in \Lambda^k V$. (Hint: Show that if $\omega \neq 0$ then there is an element $\eta \in \Lambda^{n-k}V$ such that $\omega \wedge \eta \neq 0 \in \Lambda^n V$.)

(#2). Let G be a group, and V a finite-dimensional vector space, say over $\mathbf{C}$. Recall that a linear representation of G on V is a homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V),$$

where $\mathrm{GL}(V)$ denotes the general linear group of all automorphisms of V. One says also that V is a representation of G. If we choose bases, so that an element of $\mathrm{GL}(V)$ is represented by an invertible matrix, a representation ρ amounts to giving an invertible matrix $\rho(s)$ for each $s \in G$ in such a way that $\rho(st) = \rho(s) \cdot \rho(t)$ for all $s, t \in G$.

(a). Given a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ show that for each $k > 0$, ρ induces representations

$$S^k\rho : G \longrightarrow \mathrm{GL}\big(S^k V\big) \quad \text{and} \quad \Lambda^k\rho : G \longrightarrow \mathrm{GL}\big(\Lambda^k V\big).$$

(b). Suppose that $\dim V = n$ and we realize ρ by $n \times n$ matrices. Explain concretely how we realize the matrices for $\Lambda^n \rho$.

(c). Let $V = \mathbf{C}^2$ and $G = \mathrm{GL}_2(V)$. Then there is a natural representation ρ of G on V (for which ρ is an isomorphism). Choose bases and write out explicitly the representation $S^2(\rho)$. (So for each element

$$s \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; GL_2(\mathbf{C}),$$

you should write down a 3×3 matrix in terms of a, b, c, d representing $S^2\rho(s)$.)

(#3). In this problem, V denotes a vector space of dimension n over a field k of characteristic $\neq 2$, and g is a symmetric or alternating bilinear form on V.

Let $L_g : V \longrightarrow V^*$ be the linear mapping associated to g. We define the *rank* of g to be the rank of L_g.

(a). Prove that the rank of g is $n - \dim \ker(g)$. Moreover g is non-degenerate if and only if it has rank n.

(b). Show that there exist symmetric forms of any rank $0 \leq r \leq n = \dim V$. Prove that the rank of an alternating form is even, and that there exist skew forms of any even rank $r = 2r'$ with $0 \leq r \leq n$.

(c). Let $W \subseteq V$ be a subsoace of codimension 1, and let g be a non-degenerate symmertic form on W. Denote by $g|W$ the restriction of g to W. What are the possibilities for the rank of $g|W$?

(d). A subspace $W \subseteq V$ is *isotropic* if $g|W \equiv 0$, i.e. if $g(w, w') = 0$ for all $w, w' \in W$. Find an example of a non-degenerate symmetric form on a vector space V of dimension $n = 2m$ on which there is an m-dimensional isotropic subspace. Do the same for skew forms. Find also an example of a symmetric form on an n-dimensional vector space V which has no non-trivial isotropic subspaces. (Remark: For the last part, you may want to make a judicious choice of the ground-field k.)