MATH 593
Assignment # 11
(Due Wednesday, December 11)

Hand in: #'s 2,3,4

(#1). Let V be a finite dimensional vector space over $\mathbf{R}$ (or any field of characteristic $\neq 2$), ahd let g be a non-degenerate alternating form on V. Prove that there is a basis for V with respect to which g is isomorophic to a direct sum of copies of the the form with matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(#2). Let $V = \mathbf{R}^3$ be a three-dimensional real vector space with standard basis $e_1, e_2, e_3 \in V$. Define a symmetric bilinear form on V via:

$$b(e_i, e_j) \;=\; \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

In other words, b is the symmetric form associated to the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(a). Find the rank and signature of b.

(b). Do the same for the resrtiction of b to the subspace $W \subseteq V = \mathbf{R}^3$ consisting of vectors (x, y, z) such that $x + y + z = 0$.

(#3). Let V be a finite dimensional vector space over the complex numbers $\mathbf{C}$. Recall that a *Hermitian form* on V is a function

$$h : V \times V \longrightarrow \mathbf{C}$$

which is $\mathbf{C}$-linear in the second argument, and satisfies

$$h(v, w) \;=\; \overline{h(w, v)}$$

(where the bar on the right denotes complex conjugation). Note that this implies that h is conjugate linear in the first argument, i.e. $h(a \cdot v, w) = \overline{a} \cdot h(v, w)$ for every $a \in \mathbf{C}$. We denote by $V_{\mathbf{R}}$ the real vector space underlying V (i.e. just forget that you can multiply vectors in V by complex as well as real scalars): thus $\dim_{\mathbf{R}} V_{\mathbf{R}} = 2 \cdot \dim_{\mathbf{C}} V$.

(a). Given a Hermitian form h as above, show that $h = b + \sqrt{-1} \cdot \omega$, where b is a symmetric real-valued bilinear form on $V_{\mathbf{R}}$ and ω is an alternating real-valued bilinear form on $V_{\mathbf{R}}$.

(b). Define what it should mean for h to be non-degenerate, and prove that if h is non-degenerate then so too are the forms b and ω constructed in (a).

(c). Show that if h is non-degenerate, then V has an orthogonal basis with respect to h. State and prove an analogue of Sylvester's law of interia for non-degenerate Hermitian forms.

(#4). Let $V = \mathbf{C}^n$ with its standard basis, and let h be the standard Hermitian form

$$< v, w > = \ h(v, w) \ = \ \overline{v} \cdot w.$$

An endomorphism $T : V \longrightarrow V$ is *self-adjoint* if

$$< Tu, v > = < u, Tv >$$

for all $u, v \in V$.

(a). What is the condition that T be self-adjoint in terms of the matrix of T?

(b). Show that the eigenvalues of a self-adjoint endomorphism are real.