MATH 593
Assignment # 3, Revised
(Due Monday, September 30)

Hand in: #'s 1,2,3, 4

NOTE: The first hour exam will take place in class on Monday, October 7.

(#1). Describe as concretely and efficiently as possible the fields of fractions of the following integral domains.

(a). The ring $\mathbf{Z}[x]$ of polynomials in one variable with integer coefficients.

(b). The ring $\mathbf{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbf{Q}\}$.

(c). The ring $\mathbf{C}(x)[y]$ of polynomials in a variable y whose coefficients are complex rational functions in another variable x.

(#2). Let k be a field. A *formal Laurent series* with coefficients in k is a series

$$\sum_{m=-\infty}^{\infty} a_m x^m \quad (a_m \in k)$$

such that $a_m = 0$ for all but finitely many $m < 0$. In other words, a Laurent series is like a formal power series, except that it is allowed to have finitely many terms involving negative powers of x. Denote by $k((x))$ the set of all such, so that $k((x))$ is a ring in the natural way. Show that $k((x))$ is actually a field, and in fact is (isomorphic to) the field of fractions of the ring $k[[x]]$ of formal power series. [HINT: Recall from lat week's problem set the description of units in $k[[x]]$.]

(#3). Let A be an integral domain, with field of fractions F. Recall that then given a maximal ideal $\mathfrak{m} \subset A$, one can view the localization $A_{\mathfrak{m}}$ as a subring of F. Prove that

$$A = \bigcap_{\mathfrak{m} \subset A} A_{\mathfrak{m}} \ \subset F \ ,$$

where the intersection is taken over all maximal ideals of A. [HINT: Fix $f \in \cap_{\mathfrak{m} \subset A} A_{\mathfrak{m}}$. Then for any maximal ideal $\mathfrak{m} \subset A$, there exist elements $a_{\mathfrak{m}}, b_{\mathfrak{m}} \in A$ such that $f = a_{\mathfrak{m}}/b_{\mathfrak{m}}$. It will probably be helpful to show that the ideal generated by all the $b_{\mathfrak{m}}$ is the whole ring A.]

(#4). (**Germs of continuous functions**.) This exercise will give another and more precise example of how local rings arise in the study of the local behavior of functions.

We begin by defining the notion of the germ at 0 of a continuous function on $\mathbf{R}$. Considering pairs (U, f), where $U \subset \mathbf{R}$ is an open neighborhood of $0 \in \mathbf{R}$, and $f : U \longrightarrow \mathbf{R}$ is a continuous function on U. We define an equivalence relation on the set of such pairs by decreeing that $(U, f) \sim (V, g)$ if there is an open neighborhood $W \subset U \cap V$ of 0 such that

$$f|W \ = \ g|W.$$

Denote the set of equivalence classes by $\mathcal{C}_0$; an element of $\mathcal{C}_0$ is called the *germ* at 0 of a (continuous) function. Thus two continuous functions f, g defined in neighborhoods of 0 determine the same germ at 0 iff they agree in some neighborhood of 0. In this sense, the germ of a function captures precisely the "local" data it determines in a neighborhood of 0.

Note next that $\mathcal{C}_0$ becomes a ring if we define addition and multiplication "pointwise":

$$\begin{aligned} [(U, f)] + [(V, g)] \ &= \ [(U \cap V, f + g)] \\ [(U, f)] \cdot [(V, g)] \ &= \ [(U \cap V, f \cdot g)] \end{aligned}$$

(Strictly speaking, on the right hand side we should write $[(U \cap V, f|(U \cap V) + g|(U \cap V))]$, and similarly for the second line, but this gets awfully heavy, and hopefully no harm will come from this slight abuse of notation.)

(c). Prove that $\mathcal{C}_0$ is a local ring, with maximal ideal consisting of the germs of functions that vanish at 0.

(b). As usual let $\mathcal{C}(\mathbf{R})$ denote the ring of all real-valued continuous functions on $\mathbf{R}$, let $\mathfrak{m} \subset \mathcal{C}(\mathbf{R})$ denote the maximal ideal of all functions vanishing at 0, and consider the localization $\mathcal{C}(\mathbf{R})_{\mathfrak{m}}$ of $\mathcal{C}(\mathbf{R})$ at the multiplicative subset $S = \mathcal{C}(\mathbf{R}) - \mathfrak{m}$. Show that

$$\mathcal{C}(\mathbf{R})_{\mathfrak{m}} \cong \mathcal{C}_0.$$

[NOTE: For (b), use the elementary fact that given any $\varepsilon > 0$ there exists a continuous "bump function" $\chi_\varepsilon : \mathbf{R} \longrightarrow [0, 1]$ such that $\chi_\varepsilon(t) = 1$ for $|t| < \varepsilon/2$ and $\chi_\varepsilon(t) = 0$ for $|t| > \varepsilon$.]

Remark. The notion of the germ of a function makes sense in much more general situations: one can discuss germs of continuous functions on an arbitrary topological space, germs of differentiable functions at a point in $\mathbf{R}^n$ (or on a differentiable manifold), germs of analytic functions at a point in $\mathbf{C}$ (or on a complex manifold), etc etc. [As an exercise, you might enjoy showing that the ring of germs at $0 \in \mathbf{C}$ of analytic functions is isomorphic to the ring $\mathbf{C}\{z\}$ of power series in z that converge in a neighborhood of 0.] There are similar analogues of (b).