MATH 593
Assignment # 5
(Due Friday, October 18)

Hand in: #'s 2,3,4,5

(#1) Let k be a field, and let M be the $k[x]$-module

$$M \;=\; \frac{k[x]}{(x^3+x+1)} \oplus \frac{k[x]}{(x^2)}.$$

Find explicitly a vector space V over k and an endomorphism T of V — which you should exhibit in the form of an explicit square matrix of appropriate size — such that $M \cong V_T$.

(#2). Let G be a finite group, and V a finite dimensional vector space over a field k. We denote by $GL(V)$ the group of automorphisms of V: after choosing a basis of V, $GL(V)$ can be identified with the multiplicative group $GL_n(k)$ of invertible $n \times n$ matrices with entries in k. A *reptresentation* of G on V is a homomorphism $\rho : G \longrightarrow GL(V)$. For example, if $G = S_n$ is the symmetric group on n-elements, then G has a natural representation on the n-dimensional vector space k^n via permutation matrices.

(a). Given a representation ρ of G on V, show that there is a natural way to make V into a left module over the group ring $k[G]$: call this module V_ρ.

(b). Conversely, if V has the structure of a module over $k[V]$ (in such a way that the elements $\lambda \cdot [1_G] \in k[G]$ act by scalar multiplication in on V) then there is a representation ρ of G on V such that $V \cong V_\rho$ as $k[G]$-modules.

(#3). Let k be a field, and let $f(x), g(x) \in k[x]$ be non-zero polynomials. Viewing the quotients $k[x]/(f)$ and $k[x]/(g)$ as $k[x]$-modules in the natural way, describe as explicitly as possible the $k[x]$-module

$$\mathrm{Hom}_{k[x]}\big(\, k[x]/(f)\, ,\, k[x]/(g)\, \big).$$

(The module in question is actually of the form $k[x]/(h)$ for some polynomial $h \in k[x]$. One nice solution to the problem would be to find $h(x)$.)

(#4). Let R be a ring. An R-module M is *irreducible* if the only R-submodules of M are (0) and M itself.

(a). Let $V = \mathbf{R}^2$ and let $T_\theta : V \longrightarrow V$ be rotation through θ radians. Write V_θ for V with the $\mathbf{R}[x]$-module structure determined by T_θ. For what values of θ is V_θ irreducible as an $\mathbf{R}[x]$-module?

(b). Given a representation $\rho : G \longrightarrow GL(V)$ as in (#1), what is the condition on ρ in order that the $k[G]$-module V_ρ be irreducible?

(c). Let $\rho : S_3 \longrightarrow GL_3(k)$ be the representation of the symmetric group S_3 on the three dimensional vector space $V = k^3$ given by permutation matrices. Is V irreducible as a $k[S_3]$ module?

(#5). Let R be a ring (always with 1). Given a (left) R-module M, the additive group $\mathrm{End}(M) = \mathrm{Hom}_R(M, M)$ of R-linear maps from M to itself becomes a ring in which multiplication is given by composition of endomorphisms.

(a). Assume that M is irreducible in the sense of (# 3). Show that then $\mathrm{End}_R(M)$ is a division ring, i.e. a ring in which every non-zero element is a unit.

(b). Let $V = \mathbf{R}^2$, and let $T : V \longrightarrow V$ be rotation through $\frac{\pi}{2}$ radians. Show that

$$\mathrm{End}_{\mathbf{R}[x]}(V_T) \;\cong\; \mathbf{C}$$

(as rings).