MATH 593
Assignment # 6
(Due Monday, October 28)

HAND IN: #'s 1,2,3,4

NOTE: There will be no class on Friday October 25.

(#1). Let k be a field, and suppose given an exact sequence of finite dimensonal vector spaces over k:

$$0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow 0.$$

Show that

$$\sum_{i=0}^{n} (-1)^i \dim_k A_i = 0.$$

(#2). Let R be a ring. Two short exact sequences of R-modules are said to be *equivalent* or *isomorphic* if they are related by a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_2 \arrow[r] & M_2 \arrow[r] & L_2 \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are isomorphisms. A short exact sequence

(*) $$0 \longrightarrow N \xrightarrow{u} M \xrightarrow{v} L \longrightarrow 0$$

is said to be *split* if it is equivalent to the canonical sequence

$$0 \longrightarrow N \longrightarrow N \oplus L \longrightarrow L \longrightarrow 0.$$

Prove that the following are equivalent:

(a). The sequence (*) splits;

(b). There is a homomorphism $j : M \longrightarrow N$ such that $j \circ u = 1_N$;

(c). There is a homomorphism $s : L \longrightarrow M$ such that $v \circ s = 1_L$.

(#3). Let R be a ring, and let F be a free R-module with basis $\{x_i\}_{i \in I}$. Prove the following:

(a). Given an R-module N, plus elements $n_i \in N$ $(i \in I)$, there exists a unique homomorphism: $\phi : F \longrightarrow N$ such that $\phi(x_i) = n_i$.

(b). $F \cong \bigoplus_{i\in I} R_i$, with $R_i = R$ for all i.

(c). If $M \supset N$ are R-modules such that $M/N \cong F$ then $M \cong F \oplus N$.

(#4). Let R be a ring. An R-module P is called *projective* if given any exact sequence $M \xrightarrow{u} N \longrightarrow 0$, and given any homomorphism $s : P \longrightarrow N$, there exists a homomorphism $\overline{s} : P \longrightarrow M$ such that $s = u \circ \overline{s}$:

$$\begin{tikzcd} & P \arrow[d, "s"'] & \\ M \arrow[r, "u"] & N \arrow[r] & 0. \end{tikzcd}$$

(i). Prove that a free R-module is projective.

(ii). Show that the following are equivalent:

(a). P is projective;

(b). Any exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ spits;

(c). There is a module Q such that $P \oplus Q$ is free.

(# 5). Let A be an integral domain, and let M be an A-module. Recall that an element $m \in M$ is a *torsion* element if $am = 0$ for some $0 \neq a \in A$.

(i). Prove that the subset $T(M) \subset M$ of all torsion elements is a submodule. It is called the *torsion submodule* of M.

Recall that M is *torsion-free* if $T(M) = 0$.

(ii). Prove that $M/T(M)$ is torsion free.

(iii). Let $A = \mathbf{C}[x, y]$. Give an example of a finitely generated torsion-free A-module which is not free. Give an example of an A-module M for which $T(M)$ is non-trivial proper submodule of M.