MATH 593
Assignment # 9
(Due Friday, November 22)

Hand in: All five problems.

(#1). Let $R = \mathbf{C}[x,y]$ and $I = (x,y) \subseteq R$. Show that there is a homomorphism of R-modules

$$\phi : I \otimes_R I \longrightarrow \mathbf{C}$$

determined by the formula

$$\phi(f \otimes g) \ = \ f_x(0,0)g_y(0,0) - f_y(0,0)g_x(0,0).$$

(Here f_x, f_y denote the partial derivatives of f, and similarly for g.) Deduce that

$$x \otimes y - y \otimes x \ \neq \ 0 \ \in \ I \otimes_R I.$$

(#2). Prove that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.

(#3). Let V and W be finite dimensional vector spaces over a field k. Write $V^* = \mathrm{Hom}_k(V,k)$ for the dual space of V. Prove that there is a canonical isomorphism

$$\mathrm{Hom}_k(V,W) \ \cong \ V^* \otimes_k W.$$

(#4). Let R be a commutative ring, and $I \subseteq R$ an ideal. Given any R-module M, prove that there is an isomorphism

$$M \otimes_R R/I \ \cong \ M/(IM).$$

Here $IM \subseteq M$ denotes as usual the submodule of M generated by all elements of the form rm for $r \in I$ and $m \in M$. (Hint: Consider the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ and use the right exactness of tensor product.)

(#5). Let V be a two dimensional vector space over $\mathbf{R}$, and consider the four-dimensional vector space $W = V \otimes_{\mathbf{R}} V$. Show that the set of all simple (i.e. decomposable) tensors is a quadric hypersurface in W (with respect to a suitable choice of coordinates on W).