Math 632, Lecture 3 January 12, 2004

## 1. More on stalks

Let $X$ be a topological space and $\mathcal{F}$ a presheaf on $X$ with $x \in X$. Recall the stalk at $x$ is

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

(1) Let $G$ be ab abelian group and let $\mathcal{F} = \underline{G}$. Recall we may view $\mathcal{F}$ as the sheaf of sections of the projection $G \times X \to X$ (where $G$ is given the discrete topology). We have a map $G \to \mathcal{F}(U)$ for each open $U \subseteq X$ given by $g \mapsto (u \mapsto g)$. This induces a map $G \to \mathcal{F}_x$, and this map is an isomorphism (which follows readily from the fact that elements of $\mathcal{F}(U)$ are locally constant maps to $G$).

(2) Let $X' \xrightarrow{f} X$ be a covering map and $\mathcal{F} = \Gamma_{X'/X}$ the sheaf of sections. We saw last time that there is a natural identification $\mathcal{F}_x \simeq f^{-1}(x)$. Observe that $\mathcal{F}$ is locally constant because by definition of a covering space, there exists an open covering $U_\alpha$ of $X$ such that $f^{-1}(U_\alpha) \simeq \coprod_{i \in I_\alpha} U_\alpha$, for some discrete $I_\alpha$. Therefore, $\mathcal{F}\big|_{U_\alpha}$ is the constant sheaf attached to the set $I_\alpha$.

(3) Let $X \xrightarrow{f} Y$ be a continuous map of top. spaces and $\mathcal{F}$ a sheaf on $X$. Then $f_*\mathcal{F}$ is a sheaf on $Y$ and for any $y \in Y$ we have $(f_*\mathcal{F})_y = \varinjlim_{U \ni y} \mathcal{F}(f^{-1}(U))$. If $y = f(x)$ then this direct limit is $\varinjlim_{f^{-1}(U) \ni x} \mathcal{F}(f^{-1}(U))$ which maps to $\varinjlim_{V \ni x} \mathcal{F}(V) = \mathcal{F}_x$ so we obtain a map $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$. In general, this map is not surjective.

## 2. Ringed spaces

Let $\mathscr{C}$ be a subcategory of the category of rings containing 0 and stable under formation of direct limits. We will be primarily interested in the categories of $A$-algebras (for a ring $A$) and rings.

**Definition 2.1.** A $\mathscr{C}$-ringed space $(X, \mathcal{O}_X)$ is a topological space $X$ equipped with a $\mathscr{C}$-valued sheaf $\mathcal{O}_X$.

**Definition 2.2.** A morphism of ringed spaces is a map

$$\varphi = (f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

where $f : X \to Y$ is a continuous map of top. spaces and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of $\mathscr{C}$-valued sheaves.

*Warning* 2.3. The sheaf $\mathcal{O}_X$ is not required to be a sheaf of functions with values in some fixed target and $f^\#$ is in general not determined by $f$, even though it interacts with $f$ via the maps $f^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for each open $U \subseteq Y$.

(1) Take $X = (M, \mathcal{O}_M)$ where $M$ is any $\mathbf{C}$-manifold and $\mathcal{O}_M$ the sheaf of holomorphic functions. Then for $Y = (N, \mathcal{O}_N)$ and a map $f : M \to N$ we have the induced map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ given by $f^\#_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ as $\phi \mapsto \phi \circ f$, so we get a map $\varphi = (f, f^\#)$ of ringed spaces $X \to Y$.

(2) We can take $(X, \underline{\mathbf{Z}})$ for any topological space $X$.

(3) Let $X$
*emptyset* and $\mathcal{O}_X(X) = 0$.

(4) Let $X = \{\cdot\}$ be the one point topological space, and set $\mathcal{O}_X(X) = \mathbf{C}$. Then $X$ is identified with Spec($\mathbf{C}$). One might ask what Aut($X$) is. Observe that there is a unique map $X \to X$ as a topological space, so Aut($X$) depends only on what category $\mathcal{O}_X$ takes values in. If, for example, we take the category of $\mathbf{C}$-algebras, then Aut($X$) is trivial. At the other extreme, if $\mathcal{O}_X$ has values in the category of sets then Aut($X$) is uncountable.

(5) Let $X = \text{Spec}(\mathbf{Z})$. Then the open sets are of the form $U = X - \{p_1, \ldots, p_n\}$ for some finite list of primes $p_i$. Consider the presheaf on $X$ defined by

$$\mathcal{O}_X(U) = \begin{cases} 0 & U = \emptyset \\ \mathbf{Z}[1/(p_1 \cdots p_r)] & U = X - \{p_1, \ldots, p_r\} \end{cases}.$$

The fact that we have an embedding $\mathcal{O}_X(U) \hookrightarrow \mathbf{Q}$ for all open $U$ implies that $\mathcal{O}_X$ is, in fact, a sheaf. Moreover, it is not hard to see that $\mathcal{O}_{X,(p)} = \mathbf{Z}_{(p)}$ for any prime ideal $(p)$ (in particular, $\mathcal{O}_{X,(0)} = \mathbf{Q}$).

(6) Let $X = \operatorname{Spec}(k[t]/(t^2))$. Then as a topological space, $X$ consists of a single point, $(t)$. We define $\mathcal{O}_X$ by $\mathcal{O}_X(X) = k[t]/(t)^2$.

## 3. Composition of morphisms of ringed spaces

Suppose that we have the following situation:

$$(X, \mathcal{O}_X) \xrightarrow{\varphi=(f,f^\#)} (Y, \mathcal{O}_Y) \xrightarrow{\psi=(g,g^\#)} (Z, \mathcal{O}_Z).$$

We want to understand how to form the composition $\psi \circ \varphi : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$. On the level of topological spaces, we have the maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

so we obtain $g \circ f : X \to Z$. We then want to define a map of sheaves (on $Z$)

$$\mathcal{O}_Z \to (g \circ f)_* \mathcal{O}_X$$

and we have the sheave maps $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ (both sides are sheaves on $Y$) and $g^\# : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ (now both sided are sheaves on $Z$). Recall that $(g \circ f)_* = g_* \circ f_*$ as maps of sheaves on $Z$. Moreover, by the functoriality of the pushforward map $g_*$, the map of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ gives rise to a morphism

$$g_*f^\# : g_*\mathcal{O}_Y \to g_*f_*\mathcal{O}_X = (gf)_*\mathcal{O}_X,$$

so that we have

$$\mathcal{O}_Z \xrightarrow{g^\#} g_*\mathcal{O}_Y \xrightarrow{g_*f^\#} g_*f_*\mathcal{O}_X,$$

as morphisms of sheaves on $Z$. In summary, the map $\psi \circ \varphi : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z)$ is

$$\psi \circ \varphi = (g \circ f, g_*f^\# \circ g^\#).$$

Explicitly, let $U \subseteq Z$ be open. Then we have the maps of sections

$$\mathcal{O}_Z(U) \xrightarrow{g^\#_U} \mathcal{O}_Y(g^{-1}U) \xrightarrow{f^\#_{g^{-1}U}} \mathcal{O}_X(f^{-1}g^{-1}(U)).$$

Observe that $f^\#_{g^{-1}U} = g_*f^\#_U$ and $f^{-1}g^{-1}(U) = (f \circ g)^{-1}(U)$.

*Exercise* 3.1. Check that composition thus defined is associative, and show that $\mathrm{id}_(X, \mathcal{O}_X) = (\mathrm{id}_X, \mathrm{id}_{\mathcal{O}_X})$, where $\mathrm{id}_{\mathcal{O}_X} = (\mathrm{id}_X)_*$, so that $\mathscr{C}$-ringed spaces are thus a category.

## 4. How to relate things to stalks

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two $\mathscr{C}$-ringed spaces and

$$\varphi = (f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

a morphism, given any $x \in X$ we have a map $f^\#_U : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}U)$ for all $U \subseteq Y$. This gives a map

$$\mathcal{O}_{Y,f(x)} \longrightarrow (f_*\mathcal{O}_X)_{f(x)},$$

and composing with the map $(f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$ which we defined in §1, we obtain a map

$$\varphi_x : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}.$$

## 5. Pointed $\mathscr{C}$-ringed spaces

**Definition 5.1.** A *pointed $\mathscr{C}$-ringed space* is a pair $((X, \mathcal{O}_X), x)$. A morphism of pointed $\mathscr{C}$-ringed spaces

$$\varphi = (f, f^\#) : ((X, \mathcal{O}_X), x) \longrightarrow ((Y, \mathcal{O}_Y), y)$$

is a morphism

$$(f, f^\#) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

such that $f(x) = y$.

*Exercise* 5.2. Check that the association $((X, \mathcal{O}_X), x) \longrightarrow \mathcal{O}_{X,x}$ is a functor on the category of pointed ringed spaces.

(1) Let $M, N$ be two $C^\infty$ manifolds (with $\mathcal{O}_M$ and $\mathcal{O}_N$ the sheaves of $C^\infty$ functions on $M, N$ respectively, and $\varphi = (f, f^\# : M \to N$ a $C^\infty$ map, where $f^\#$ is "composition with $f$," that is $f^\# : \mathcal{O}_{N,n} \to \mathcal{O}_{M,m}$ is given on sections over an open $U$ by $\alpha \to \alpha \circ f$.

Suppose that $m \in M$ and $n = f(m)$. Then we obtain a map

$$\varphi_m : \mathcal{O}_{N,n} \longrightarrow \mathcal{O}_{M,m}$$

induced by composition with $f$. Observe that $\mathcal{O}_{N,n}$ is a local ring with maximal ideal $\mathfrak{m}_n$ the ideal of all functions vanishing at $n$. If $\alpha(n) = 0$ then $\alpha \circ f(m) = 0$ so that $\alpha \circ f \in \mathfrak{m}_m$, where $\mathfrak{m}_m$ is the unique maximal ideal of $\mathcal{O}_{M,m}$ consisting of functions vanishing at $m$. Thus, $\varphi_m(\mathfrak{m}_n) \subseteq \mathfrak{m}_m$, so that $\varphi_m$ is a local map of local rings and gives rise to a morphism of pointed ringed spaces. Observe that all the stalks are local rings.

(2) For an example of a ringed space where none of the stalks are local rings, consider the ringed space $(X, \underline{\mathbf{Z}})$, where $X$ is any topological space. The stalk at any point $\underline{\mathbf{Z}}_x$ is just $\mathbf{Z}$, which is not local.