MATH 632, LECTURE 17 FEBRUARY 16, 2004

1. Base change

Let $f : X \to S$ and $\pi : S' \to S$ be schemes. Then we have the cartesian diagram

$$\begin{CD} X' = X \times_S S' @>>> X \\ @VVV @VV{f}V \\ S' @>{\pi}>> S \end{CD}$$

and we say that X' is the *base change* of X by π. This construction generalizes extension of scalars, which is just the case where X, S, S' are affine. As an example, consider

$$\begin{CD} X_s = X \times_S \operatorname{Spec} \kappa(s) @>>> X \\ @V{f_s}VV @VV{f}V \\ \operatorname{Spec} \kappa(s) @>>> S \end{CD}$$

and suppose that f is locally of finite type. Then one can show that X_s is a locally finite type $\kappa(s)$-scheme.

Recall that for any ring A we defined $\mathbf{P}^n_A = \operatorname{Proj}(A[T_0, \ldots, T_n])$. Then we have $\mathbf{P}^n_A \simeq \mathbf{P}^n_{\mathbf{Z}} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} A$. More generally, if S is any $\mathbf{N}$-graded ring and A is an S_0-algebra then $S \otimes_{S_0} A$ has a natural $\mathbf{N}$-grading given by $(S \otimes_{S_0} A)_d = S_d \otimes_{S_0} A$ and we have, by the isomorphism $S_(f) \otimes_{S_0} A \simeq (S \otimes_{S_0} A)_{f \otimes 1}$,

$$\begin{CD} \operatorname{Proj}(S \otimes_{S_0} A) @>>> \operatorname{Proj}(S) \\ @VVV @VVV \\ \operatorname{Spec} A @>>> \operatorname{Spec} S_0 \end{CD}$$

As an example, let $g_j \in A[T_0, \ldots, T_n]$ be homogenous polynomials and consider the closed immersion

$$\operatorname{Proj}(A[T_0, \ldots, T_n]/(g_1, \ldots, g_r)) \hookrightarrow \mathbf{P}^n_A.$$

The fiber over a point $\operatorname{Spec} \kappa(x) \to \operatorname{Spec} A$ is

$$\operatorname{Proj}(\kappa(x)[T_0, \ldots, T_n]/(g_1(x), \ldots, g_r(x))),$$

where $g_j(x)$ denotes the image of g_j in $k(x)[T_0, \ldots, T_n]$.

(1) Consider

$$\begin{array}{ccc} \operatorname{Spec} \mathbf{C}[\lambda, x, y]/(y^2 - x(x-1)(x-\lambda)) & \longrightarrow & \mathbf{A}^2_{\mathbf{C}[\lambda]} \\ & \searrow & \downarrow \\ & & \operatorname{Spec} \mathbf{C}[\lambda] = \mathbf{A}^1_{\mathbf{C}} \end{array}$$

The fiber over $\lambda_0 \in \mathbf{C}$ is $\operatorname{Spec} \mathbf{C}[x, y]/(y^2 - x(x-1)(x-\lambda_0))$.

(2) Now consider $\operatorname{Spec} k[x, y]/(y^2 - x) \to \operatorname{Spec} k[x]$ and let $x_0 \in k$. The fiber over x_0 is $\operatorname{Spec} k[y]/(y^2 - x_0)$, which consists of 2 points if $\operatorname{char} k \neq 2$ and $x_0 \neq 0$ and which is the nonreduced scheme $\operatorname{Spec} k[y]/y^2$ if $x_0 = 0$.

(3) Suppose that $K/\mathbf{Q}$ is a number field and $\mathcal{O}$ its ring of integers. Let $x = \{p\} \in \operatorname{Spec} \mathbf{Z}$ and consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O} & \longleftarrow & \operatorname{Spec} \mathcal{O}/p\mathcal{O} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{Z} & \longleftarrow & \operatorname{Spec} \kappa(x) \end{array}$$

Here $\mathcal{O}/p\mathcal{O}$ is a finite dimensional $\kappa(x)$-algebra

(4) From the point of view of base change, we can reduce a scheme over $\operatorname{Spec} A$ modulo an ideal $I \subseteq A$ by taking the fiber product with $\operatorname{Spec} A/I$ over $\operatorname{Spec} A$. One might be interested in understanding a scheme A over $\mathbf{Q}$ "modulo p." Concretely, this amounts to the existence of a "nice model" $\mathcal{A}$ over $\operatorname{Spec} \mathbf{Z}$ with the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbf{Q} & \longrightarrow & \operatorname{Spec} \mathbf{Z} \end{array}$$

2. Behavior of properties of morphisms under base change

Definition 2.1. A map of schemes $f : X \to Y$ is *flat* if for all $x \in X$ the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Lemma 2.2. *If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine then $f : X \to Y$ is flat if and only if A is a flat B-algebra.*

Proof. By commutative algebra, the map $\varphi : B \to A$ is flat if and only if the map $B \to A_{\mathfrak{p}}$ is flat for all $\mathfrak{p} \in X$. But this map factors through $B_{\varphi^{-1}(\mathfrak{p})}$ and is flat if and only if the induced map $\mathcal{O}_{Y,f(x)} = B_{\varphi^{-1}(\mathfrak{p})} \to A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is flat. ■

Definition 2.3. A scheme is *regular* if it is locally Noetherian and all local rings are regular.

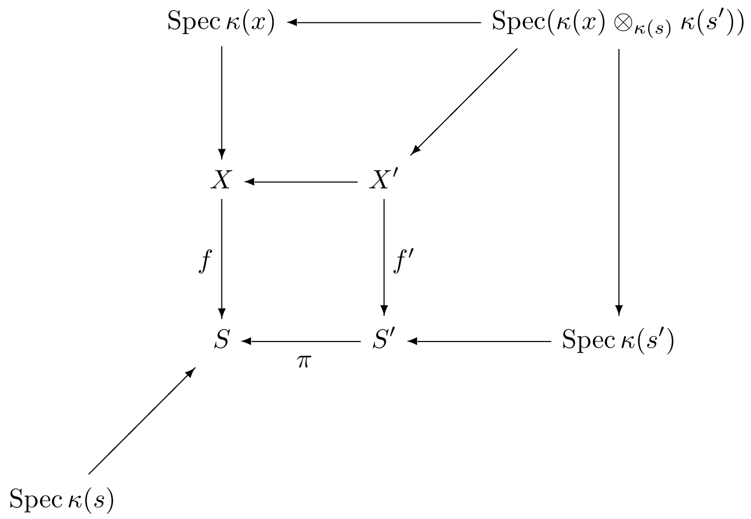
By a famous Theorem of Serre, if A is a regular local ring then so is $A_{\mathfrak{p}}$ for any prime $\mathfrak{p} \subseteq A$.

As a warning, we remark that base change does not necessarily preserve all "nice" properties of a scheme. For example, it can happen that for a regular scheme X of finite type over a field k, the base change to k'/k is *not* regular. For example, one might take k'/k a purely inseparable extension of degree greater than 1. Then $k' \otimes_k k'$ is not reduced, even though k' is.

However, many nice properties of morphisms are preserved by base change. Let X, S' be schemes over a scheme S and let $X' = X \times_S S'$. Then we have the cartesian diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \underset{\pi}{\longleftarrow} & S' \end{array}$$

and we claim that for the following list of properties P, the morphism f has property P implies that the morphism f' has the property P: Open immersion, closed immersion, affine, finite type and locally finite type, quasi-compact, flatness, surjective. All of these facts are easy to show, with perhaps the exception of surjectivity. Let $s' \in S'$ and put $s = \pi(s')$. Since the map $X \twoheadrightarrow S$ is surjective, we can find $x \in X$ mapping to s. This gives the diagram:



and since $\kappa(x) \otimes_{\kappa(s)} \kappa(s')$ is *nonzero*, this gives at least one point of X' mapping to s', as required. The same argument also shows that $s' \in S'$ is in $f'(X')$ if and only if $\pi(s') = s \in S$ lies in $f(X)$. We remark that injectivity is usually destroyed by base change. Indeed, let L/K be an extension of fields. Then $\operatorname{Spec} L \to \operatorname{Spec} K$ remains injective after arbitrary base change if and only if L/K is purely inseparable.

Definition 2.4. A map $f : X \to S$ of schemes is *universally injective* (also called radical) if it is injective after arbitrary base change.

One can show that a map f is radical if and only if f is injective and for all $x \in X$ the extension $\kappa(x)/\kappa(f(x))$ is purely inseparable. This is a very restrictive condition indeed.