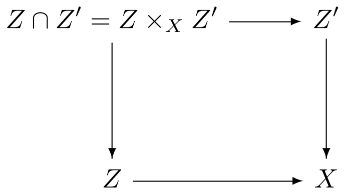
Math 632, Lecture 18 February 18, 2004

1. Scheme theoretic intersection

If $i : Z \to X$ and $i' : Z' \to X$ are closed immersions of ringed spaces and there exists a compatible map $f : Z \to Z'$ then f is also a closed immersion. If in addition there exists a compatible $f' : Z' \to Z$ then necessarily $f \circ f' = \mathrm{id}_{Z'}$ and $f' \circ f = \mathrm{id}_Z$.

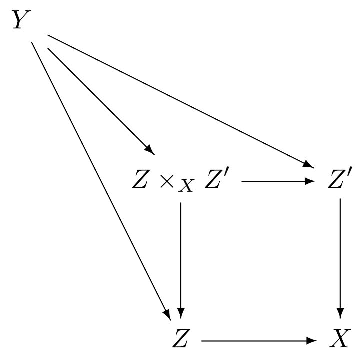
We say the two closed immersions $Z, Z' \to X$ are *equivalent* if there exists a compatible isomorphism $Z \simeq Z'$. Such a map is unique if it exists.

Definition 1.1. Let $Z, Z' \to X$ be closed immersions of schemes. The *scheme theoretic intersection* of Z, Z' is



and $Z \cap Z'$ is realized, via either projection, as a closed subscheme of X whose image is contained in Z and Z'.

Observe that if $Y \to X$ is a closed subscheme contained in Z and Z' then we get the diagram



by the universal property of the fiber product $Z \times_X Z'$, so that Y is contained in $Z \cap Z'$.

Proposition 1.2. *The topological space $|Z \cap Z'|$ of $Z \cap Z'$ is $|Z| \cap |Z'|$.*

Proof. Let $U = X - Z \times_X Z'$ and observe that U is open. Observe that $U = (X - Z) \cup (X - Z')$. Indeed, $U, (X - Z), (X - Z')$ are open subschemes of X and $f : Y \to X$ factors through U if and only if for all $y \in Y$ we have $f(y) \notin Z \times_X Z'$, that is, if and only if the map $\mathrm{Spec}\,\kappa(f(y)) \to Y$ fails to factor through Z and Z' (using the universal property of the fiber product). Since $\mathrm{Spec}\,\kappa(f(y))$ is reduced, such factorization fails if and only if $f(y) \notin Z$ or $f(y) \notin Z'$, *i.e.* if and only if $f(y) \in (X - Z) \cup (X - Z')$. Thus, $f : Y \to X$ factors through U if and only if it factors through $(X - Z) \cup (X - Z')$, so the two opens are uniquely isomorphic as they have the same universal property. ∎

We now ask: what is the ideal sheaf $\mathcal{I}_{Z \cap Z'} \subseteq \mathcal{O}_X$?

Proposition 1.3. *We have $\mathcal{I}_{Z \cap Z'} = \mathcal{I}_Z + \mathcal{I}_{Z'}$ inside $\mathcal{O}_X$, where for any two sheaves $\mathcal{F}, \mathcal{G}$ on a space Y the sum $\mathcal{F} + \mathcal{G}$ is the sheaf image of $\mathcal{F} \oplus \mathcal{G}$ under the addition map.*

Proof. Everything is local on X, so we may assume $X = \mathrm{Spec}\,A$ and $Z = \mathrm{Spec}\,A/I$ and $Z' = \mathrm{Spec}\,A/I'$. Let us compute stalks at $x \in X$. We have

$$Z \times_X Z' = \mathrm{Spec}(A/I \otimes_A A/I') = \mathrm{Spec}\,A/(I + I')$$

and

$$\mathcal{I}_{Z \cap Z'} = (I + I')_{\mathfrak{p}} = I_{\mathfrak{p}} + I'_{\mathfrak{p}}$$

in $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$. Thus we have $\mathcal{I}_{Z \cap Z', x} = \mathcal{I}_{Z,x} + \mathcal{I}_{Z',x} = (\mathcal{I}_Z + \mathcal{I}_{Z'})_x$. ∎

As an example, suppose that $\operatorname{char} k \neq 2$ and put $C_i = \operatorname{Spec} k[x,y]/(y + (-1)^i x^n)$ for $i = 1, 2$. Then

$$C_1 \cap C_2 = \operatorname{Spec} k[x,y]/(y - x^n, y + x^n) = \operatorname{Spec} k[x]/x^n \hookrightarrow \operatorname{Spec} k[x].$$

2. Remarks on base change

If k is a field and X a finite type k-scheme, it is important to study behavior of properties of morphisms and of X with respect to extension of k. So for example, if k'/k is an extension and $X' = X \otimes_k k' = X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$, then X being irreducible does *not* imply that X' is, and similarly for reduced.

We would like to descend properties. That is, we wish to know that if a property holds after base change, then it held before base change. It will turn out that if $\pi : S' \to S$ is any base change that is *faithfully flat* (i.e. flat and surjective) then most "nice" properties will hold after base change if and only if they held before base change. A useful principle for finite type schemes over a field k is that we can check many properties over any algebraically closed extension of a field k. We will see this later.

Flat maps will turn out to have especially nice properties. For example, let $f : X \to Y$ be a flat map of locally noetherian schemes and suppose $f(x) = y$. Then $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X_y,x}$. The key point here is that $\mathcal{O}_{X_y,x} \simeq \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$.

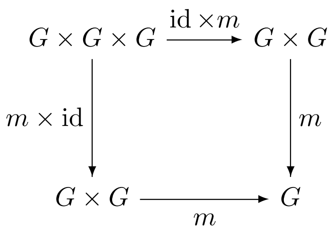
3. Group schemes

Let $\mathscr{C}$ be the category of sets and $\{\cdot\}$ the final object. A *group object* G is an object G together with a triple of morphisms:

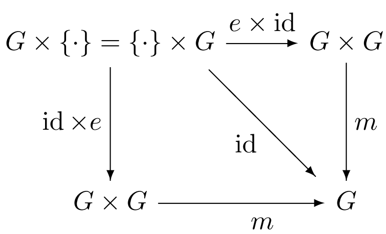
$$\left(G, G \times G \xrightarrow{m} G, G \xrightarrow{i} G, \{\cdot\} \xrightarrow{e} G\right)$$

such that the following diagrams commute:

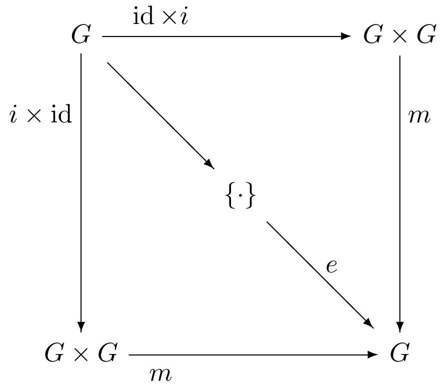
(1) Associativity:



(2) Identity:



(3) Inverse:



It is clear that in any category with finite products and a final object we have a notion of a group object. For example, let $\mathscr{C}$ be the category of S-schemes. Here the final object is S and products are fiber products over S. So an S-group G is a scheme $\pi : G \to S$ together with a section $e : S \to G$ such that $\pi \circ e = e \circ \pi = \mathrm{id}$ and maps $m : G \times_S G \to G$ and $i : G \to G$ making the above three diagrams commute.

As an example, consider $\mathbf{G}_a = \mathbf{A}^1_{\mathbf{Z}}$. We have $\mathbf{G}_a(Y) = \mathrm{Hom}(Y, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(Y, \mathcal{O}_Y)$, which is an additive group, and this is natural in Y, *i.e.* a map $Y' \to Y$ gives a map of groups $\mathbf{G}_a(Y) \to \mathbf{G}_a(Y')$. It will follow by Yoneda's lemma, which we will see later, that $\mathbf{G}_a$ is a group scheme.

Another important example is $\mathrm{GL}_n = \mathrm{Spec}(\mathbf{Z}[X_i]_{(}\det))$, and $\mathrm{GL}_n(Y)$ is the set of $n \times n$ matrices having entries in $\Gamma(Y, \mathcal{O}_Y)$ with determinant in $\Gamma(Y, \mathcal{O}_Y^{\times})$, functorially in Y as a group via multiplication. The simplest case is $\mathbf{G}_m = \mathrm{GL}_1 = \operatorname{Spec} \mathbf{A}^1_{\mathbf{Z}} - \{0\} = \operatorname{Spec} \mathbf{Z}[X, X^{-1}]$. Here, $\mathbf{G}_m(Y) = \Gamma(Y, \mathcal{O}_Y^{\times})$. The map $\mathbf{G}_m(Y) \to \mathbf{G}_m(Y)$ given by $u \mapsto u^n$ has kernel $\mu_n(Y)$, the Y-points of $\mu_n = \operatorname{Spec} \mathbf{Z}[X]/(X^n - 1)$. This is a closed subscheme of $\mathbf{G}_m$. We can even consider the $\mu_p/\mathbf{F}_p$, that is, the fiber of $\mu_p \to \operatorname{Spec} \mathbf{Z}$ over p. This is naturally a group scheme, $\operatorname{Spec} \mathbf{F}_p[T]/(T^p - 1)$ whose Y points are $\mu_p(Y) = \{u \in \Gamma(Y, \mathcal{O}_Y^{\times}) : u^p = 1\}$.