Math 632, Lecture 19 February 20, 2004

1. The diagonal and immersions

Definition 1.1. A subset $T \subseteq X$ of a topological space is *locally closed* if there exists an open $U \subseteq X$ containing T such that T is closed in U.

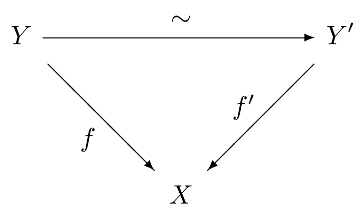
Definition 1.2. A morphism $f : Y \to X$ of locally ringed spaces is an *immersion* if f is a homeomorphism onto a locally closed subset of X and $f^\# : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is surjective for all y.

Thus we see immediately that open immersions and closed immersions are immersions. Similarly, any quasi-projective variety X over k comes equipped with an immersion $X \to \mathbf{P}^n_k$.

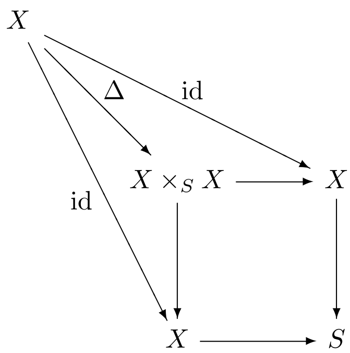
Theorem 1.3. (1) *Let $f : Y \to X$ be a morphism of locally ringed spaces and $\{U_i\}$ an open cover of X (equivalently of* $\operatorname{im} f$*). Then f is an immersion if and only if $f : f^{-1}(U_i) \to U_i$ are immersions for all i.*
(2) *A composite of immersions is an immersion.*
(3) *If $f : Y \to X$ is an immersion and $X' \to X$ any base change, then $f' : Y \times_X X' \to X'$ is an immersion as well.*
(4) *Suppose that $f : Y \to X$ is an immersion. Then f is a closed immersion if and only if $f(Y)$ is closed. SImilarly, f is an open immersion if and only if $f(Y)$ is open and $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are isomorphisms (equivalently injective) for all $y \in Y$*

Proof. Left as an exercise. ∎

Definition 1.4. A *subscheme* of X is an immersion $f : Y \to X$ up to the equivalence relation $Y \sim Y'$ if and only if there exists a commutative diagram



Definition 1.5. Let $X \to S$ be a morphism. The *diagonal* morphism $\Delta_{X|S}$, or simply $\Delta : X \to X \times_S X$, is the morphism induced by the diagram



When $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$ are both affine, we obtain the map $\Delta^* : A \otimes_B A \to A$. Since we must have $\Delta^*(1 \otimes a) = a = \Delta^*(a \otimes 1)$ (by virtue of the fact that the maps $X \to X$ inducing the diagonal are the identity maps), and hence that $\Delta^*(a_1 \otimes a_2) = a_1 a_2$.

Lemma 1.6. *The diagonal $\Delta_{X|S} : X|rightarrowX \times_S X$ is an immersion.*

Proof. First of all, the property of being an immersion is a local property, and since an open cover of S pulls back to an open cover of $X \times_S X$, it suffices to treat the case of affine S. Similarly, as in a previous lecture, we may assume that X is affine. But in the case of X, S affine, the lemma is clear. ∎

Corollary 1.7. *The diagonal $\Delta_{X|S}$ is a closed immersion if and only if* $\operatorname{im} \Delta$ *is closed.*

Definition 1.8. A morphism $X \to S$ is *separated* if $\operatorname{im} \Delta_{X|S}$ is closed.

We observe that this condition is the right scheme-theoretic analog of the Hausdorff condition as for a topological space X, the diagonal $X \to X \times X$ has closed image if and only if X is Hausdorff.

Remark 1.9. Observe that a map from T to the diagonal (*i.e.* the image of $\Delta_{X|S} : X \to X \times_S X$ is a map to $X \times_S X$ that factors through X. Such maps are just the image of the diagonal set map $X(T) \to X(T) \times X(T)$.

Suppose we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \qquad \swarrow & \\ & S & \end{array}$$

We can form the *graph* morphism $\Gamma_f : X \to X \times_S Y$: it is the morphism induced by the following diagram:

$$\begin{array}{ccccc} X & & & & \\ & \overset{\Gamma_f}{\searrow} & & \overset{f}{\searrow} & \\ & & X \times_S Y & \longrightarrow & Y \\ & \underset{\text{id}}{\searrow} & \downarrow & & \downarrow \\ & & X & \longrightarrow & S \end{array}$$

We claim that Γ_f is an immersion. Indeed, by Theorem 1.3, it suffices to show that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_S Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1} \\ Y & \xrightarrow{\Delta_{Y|S}} & Y \times_S Y \end{array}$$

is cartesian. To check that it is indeed cartesian, we observe that to give maps $T \overset{(\phi_1, \phi_1')}{\longrightarrow} X \times_S Y$ and $T \overset{\phi_2}{\longrightarrow} Y$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{(\phi_1, \phi_1')} & X \times_S Y \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle f \times 1} \\ Y & \xrightarrow[1 \times 1]{\Delta_{Y|S}} & Y \times_S Y \end{array}$$

commutes requires that we have $(\phi_2, \phi_2) = (f\phi_1, \phi_1')$, i.e. that $\phi_1' = \phi_2$ and $f\phi_1 = \phi_2$, so that the map $\phi_1 : T \to X$ determines ϕ_1' and ϕ_2. Thus, to give such maps is to give a map $T \to X$, whence the cartesian claim follows.

Now put $U_i = \mathbf{A}^1$ for $i = 1, 2$ and let X be the scheme obtained by glueing U_1, U_2 along $\mathbf{A}^1 - \{0\}$, so that $U_1 \cap U_2 = \mathbf{A}^1 - \{0\}$. We obtain the diagonal $X \to X \times_S X$ whose image is covered by $U_i \times U_j$, and Δ is a closed immersion if and only if each of the induced morphisms $\Delta : \text{id}^{-1}(U_i \times U_j) \to U_i \times U_j$ is a closed immersion. But as $\text{id}^{-1}(U_1 \times U_2) = U_1 \cap U_2$, this entails that the map $U_1 \cap U_2 \overset{(\text{id},\text{id})}{\to} U_1 \times U_2$ be a closed immersion. But this is a morphism $\mathbf{A}^1 - \{0\} \to \mathbf{A}^1 \times \mathbf{A}^1 = \mathbf{A}^2$, whose image is the line $t_1 = t_2$ (where t_i are coordinates on the two copies of $\mathbf{A}^1$) with the origin deleted, and hence is not closed. Thus $X \to S$ is not separated.