Bryden Cais Kris Klosin
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1. Introduction

The goal of these notes is to show that the modular jacobian $J_0(N)_{/\mathbf{Q}}$ has a nontrivial quotient of rank 0 for any prime N such that the genus of $X_0(N)$ is positive.

Recall the following theorem from James' talk:

Theorem 1.1. *Let $A_{/\mathbf{Q}}$ be an abelian variety with good reduction outside of N and purely toric reduction at N. Suppose moreover that $\mathcal{A}[p]$ is admissible for some $p \neq N$, where $\mathcal{A}_{/\mathbf{Z}}$ is the Néron model of A. Then $A_{/\mathbf{Q}}$ has rank 0.*

Our aim is to construct a nonzero isogeny factor $J_{\mathfrak{P}}$ of $J_0(N)$ satisfying the hypotheses of this theorem. Before we construct $J_{\mathfrak{P}}$ however, we will show that the above reduction properties of A are inherited by any isogeny factor, and will discuss the relationship between isogenies $A \sim A' \times A''$ (with A, A', A'' abelian varieties over a field K) and idempotents in the ring of endomorphisms $\mathrm{End}^0(A)$ in the isogeny cateogory of abelian varieties over K.

2. Semi-abelian reduction

In order to apply Theorem 1.1 to the isogeny factor $J_{\mathfrak{P}}$ of $J_0(N)$ that we will construct, we must show that $J_{\mathfrak{P}}$ has good reduction outside N and purely toric reduction at N, and that $J_{\mathfrak{P}}[p]$ is admissible. We first show that all isogeny factors of $J_{\mathfrak{P}}$ inherit these reduction properties.

Let K be the fraction field of a discrete valuation ring $\mathcal{O}$ with normalized valuation v and residue field k. Fix a separable closure K_s of K and a prime v_s of K_s lying over v, and denote by I the inertia group of v_s. Let A be an abelian variety over K, and denote by $\mathcal{A}$ its Neron model over $\mathcal{O}$, and by $\mathcal{A}^0$ the connected component of the identity of $\mathcal{A}$. Recall that we say that A has *semiabelian reduction* with respect to $\mathcal{O}$ if the reduction $\mathcal{A}^0_s$ is an extension of an abelian variety by a torus.

The aim of this section is to prove the following propositon:

Proposition 2.1. *Let the notation be as above. Suppose moreover that A is isogenous to the product $A' \times A''$ of abelian varieties A' and A'' over K. Then if A has good (resp. semi-abelian) reduction, so do A' and A''.*

Proof. We will only prove the case of semi-abelian reduction as the other cases are easier. Choose a prime $\ell \neq \operatorname{char} k$. Let $V_\ell(A) := T_\ell(A) \otimes \mathbf{Q}$ denote the rational ℓ-adic Tate module of A. First observe that

$$V_\ell(A) = V_\ell(A' \times A'') = V_\ell(A') \times V_\ell(A'').$$

Grothendieck's criterion for semi-abelian reduction (cf. [1], theorem 6, page 184) asserts that A has semiabelian reduction if and only if there exists a subspace V of $V_\ell(A)$ stable under the inertia group I such that I acts trivially on both V and the quotient $V_\ell(A)/V$. (cf. [1, Theorem 6, p. 184]) Note that V can (and will) be taken to be the maximal subspace of $V_\ell(A)$ on which the action of I is trivial.

Let V', V'' be the maximal subspaces of $V_\ell(A')$ and $V_\ell(A'')$ respectively, on which I operates trivially. Obviously $V' \times V'' = V$. Moreover, as $(V_\ell(A')/V') \times (V_\ell(A'')/V'') = V_\ell(A)/V$, we conclude that I acts trivially on $V_\ell(A')/V'$ and on $V_\ell(A'')/V''$. This finishes the proof. ∎

Remark 2.2. Let us prove that if A has toric reduction, then so do A' and A''. Assume that A has toric reduction. First note that since A and $A' \times A''$ are isogenous, they are both semi-abelian as their rational Tate modules are isomorphic. Denote by ϕ_K an isogeny $A \to A' \times A''$. Then there exists an isogeny $\psi_K : A' \times A'' \to A$ and a nonzero integer n, such that $\phi_K \psi_K = n_{A' \times A''}$ and $\psi_K \phi_K = n_A$ (cf.[1], p. 169). The Neron mapping property ensures that ϕ_K and ψ_K extend to morphisms ϕ and ψ between $\mathcal{A}$ and the Neron model $\mathcal{A}' \times \mathcal{A}''$ of $A' \times A''$. We have $\mathrm{Ner}(A' \times A'') = \mathcal{A}' \times \mathcal{A}''$ because of the Neron mapping property: if Z is a smooth $\mathcal{O}$-scheme, then

$$\begin{aligned}\mathrm{Hom}_K(Z_K, A' \times A'') &= \mathrm{Hom}_K(Z_K, A') \times \mathrm{Hom}_K(Z_K, A'') \\ &= \mathrm{Hom}_{\mathcal{O}}(Z, \mathcal{A}') \times \mathrm{Hom}_{\mathcal{O}}(Z, \mathcal{A}'') = \mathrm{Hom}_{\mathcal{O}}(Z, \mathcal{A}' \times \mathcal{A}'').\end{aligned}$$

We will first show that $\bar{\phi} : \bar{\mathcal{A}} \to \bar{\mathcal{A}}' \times \bar{\mathcal{A}}''$ is an isogeny. First note that $\bar{\psi}\bar{\phi} = \bar{\psi\phi} = [n]$. Since A is semi-abelian, $\bar{\mathcal{A}}^0$ is an extension of an abelian variety by a torus. As multiplication by a nonzero integer is surjective on abelian varieties and on tori, we conclude that $[n]$ is surjective on $\bar{A}^0$, hence an isogeny. Since $[n] = \bar{\psi}\bar{\phi}$, and $\bar{A}$ and $\bar{A}' \times \bar{A}''$ have the same dimension, we conclude that $\bar{\phi}$ is surjective on the identity components and has finite kernel, hence is an isogeny.

The isogeny between $\bar{A}$ and $\bar{A}' \times \bar{A}''$ induces an isogeny between $\bar{A}^0$ and $(\bar{A}' \times \bar{A}'')^0 = \bar{A}'^0 \times \bar{A}''^0$, so it remains to check that for connected commutative smooth algebraic groups over a field k, the property of being a torus is preserved under isogeny and under formation of direct factors. This is clear since by the structure theorem for commutative algebraic groups over $\bar{k}$ such a group G is a torus if and only if G is affine and $\dim(V_\ell(G)) = \dim(G)$, where $l \neq \operatorname{char} k$ is a prime.

3. Idempotents inside the endomorphism ring of abelian varieties

In this section we remark upon some generalities concerning the relationship between idempotents in the ring $\operatorname{End}^0(A)$ for an abelian variety A and isogenies $A \sim A' \times A''$ with A', A'' abelian varieties.

Proposition 3.1. *Let A be an abelian variety over a field K. There is a one-to-one correspondence between ordered pairs (A', A'') of abelian subvarieties of A such that $A' \times_K A'' \to A$ is an isogeny and idempotents in the ring $\operatorname{End}^0(A)$; the operation $(A', A'') \mapsto (A'', A')$ corresponds to $e \mapsto 1 - e$.*

Proof. Suppose we are given a pair (A', A'') of abelian subvarieties of A such that the natural map $\phi : A' \times A'' \to A$ is an isogeny. Let ϕ^{-1} denote the inverse of ϕ inside $\operatorname{Hom}^0(A, A' \times A'')$. Consider the composition

$$e : A \xrightarrow{\phi^{-1}} A' \times A'' \xrightarrow{\mathrm{pr}} A' \overset{\iota}{\hookrightarrow} A' \times A'' \xrightarrow{\phi} A,$$

where ι is inclusion and pr is projection. Note that e is an idempotent in $\operatorname{End}^0(A)$, and by definition of ϕ, it lifts the identity on A'.

Conversely, given an idempotent $e \in \operatorname{End}^0(A)$, choose a nonzero integer n such that $ne \in \operatorname{End}(A)$, and set A' to be the image of ne. Note that A' is independent of n, since multiplication by a nonzero integer is surjective on an abelian variety. The Poincare Reducibilty Theorem (c.f.[4]) guarantees the existence of a unique abelian subvariety A'' of A, such that $A' \times A'' \to A$ is an isogeny (the map being addition). One checks that the two processes are inverses of each other.

■

4. The optimal quotient $J_{\mathfrak{P}}$ of $J_0(N)$

In this section we will construct a nonzero optimal quotient $J_{\mathfrak{P}}$ of $J_0(N)$ such that the action of $\mathbf{T}$ on $J_0(N)$ induces an action on $J_{\mathfrak{P}}$.

Fix N and let $\mathbf{T}$ denote the Hecke ring. Let $\mathscr{I} \subseteq \mathbf{T}$ be the Eisenstein ideal let $\mathfrak{P} \supseteq \mathscr{I}$ be a prime of residual characteristic $p \neq N$ (we have seen that such a $\mathfrak{P}$ exists). Observe that $\mathfrak{P}_p := \mathfrak{P}(\mathbf{T} \otimes \mathbf{Z}_p)$ is a prime of $T \otimes \mathbf{Z}_p$.

Consider the subring $\mathbf{T} \otimes_{\mathbf{Z}} \mathbf{Q} \subseteq \operatorname{End}^0(J_0(N))$ (we have seen that in fact equailty holds, but we do not use this). By Proposition 3.1, the decomposition of $\mathbf{T} \otimes \mathbf{Q}$ into a product of fields gives us the isogeny decomposition of $J_0(N)$ as a product of $\mathbf{Q}$-simple abelian subvarieties:

$$J_0(N) \sim \prod_{\mathfrak{q} \in \operatorname{MinSpec}(\mathbf{T})} J_{\mathfrak{q}}. \tag{1}$$

We put

$$\tilde{J}_{\mathfrak{P}} = \prod_{\substack{\mathfrak{q} \in \operatorname{MinSpec}(\mathbf{T}) \\ \mathfrak{q} \subseteq \mathfrak{P}}} J_{\mathfrak{q}}.$$

Definition 4.1. Let A, A', and A'' be abelian varieties and suppose that A' and A'' are quotients of A. We say that A' and A'' are *isogenous as quotients* of A if there exists an isomorphism $\phi : A' \to A''$ in the isogeny category commuting (in that category) with the quotient maps $A \to A'$ and $A \to A''$.

Claim 4.2. *Let $A \to A''$ be a surjective map of abelian varieties over a field K. Then there is a unique quotient abelian variety A_{opt}, isogenous to A'' as a quotient of A, that is the quotient of A by an abelian subvariety.*

Proof. We treat existence first. By the Poincaré Reducibility Theorem over K, we see that A is isogenous to a product of simple abelian subvarieties, say $A \leftarrow \prod_{i=1}^n A_i$, and since A'' is a quotient of A we have (renumbering if necessary) an isogeny $A'' \sim \prod_{i=1}^k A_i$, as quotients of A, for some $k \leq n$. Define $A' = \prod_{k<i\leq n} A_i$. Then we have a morphism

$$A' \overset{\iota}{\hookrightarrow} \prod_{i=1}^{n} A_i \xrightarrow{\varphi} A,$$

where φ is an isogeny. Observe that $\varphi \circ \iota(A')$ is an abelian subvariety of A. We set

$$A_{\mathrm{opt}} = A/\varphi \circ \iota(A').$$

It is clear that A_{opt} is the quotient of A by an abelian subvariety and it follows (again from Poincaré Reducibility) that A_{opt} is isogenous to A'' as a quotient of A.

We claim that A_{opt} has the following universal property: if $q'' : A \to \tilde{A}$ is any quotient isogenous to the quotient $q : A \to A_{\mathrm{opt}}$ in the sense of Definition 4.1 then q'' factors uniquely through q in the isogeny category. Indeed, if $\tilde{A}$ and A_{opt} are isogenous as quotients, there exists an isomorphism (In the isogeny category) $\phi : A_{\mathrm{opt}} \to \tilde{A}$. By composing ϕ with multiplication by a large enough integer, we obtain *honest* maps $n \circ q'' : A \to \tilde{A}$ and $\psi = n \circ \phi : A_{\mathrm{opt}} \to A$ such that $\psi \circ q = n \circ q''$. Hence $n \circ q''(\ker q) = 0$. But multiplication by n is an isogeny, and hence has finite kernel, while $q''(\ker q)$ is connected (as $\ker q$ is an abelian subvariety of A), so that $q''(\ker q) = 0$.

It follows that the map $A \to A''$ factors uniquely through A_{opt}. From this universal property of A_{opt}, we see at once that A_{opt} is unique up to unique isomorphism. ∎

Applying the above claim to $A = J_0(N)$ and the isogeny factor $\tilde{J}_{\mathfrak{P}}$, we obtain an optimal quotient $J_{\mathfrak{P}}$ of $J_0(N)$. We claim that $\mathbf{T}$ acts on $J_{\mathfrak{P}}$. This will follow from the following more general theorem:

Theorem 4.3. *Let $\pi : A \to A'$ be a surjective map of abelian varieties over a field K of characteristic 0 having an abelian variety kernel B and let $T \in \mathrm{End}(A)$. Assume there exists $T' \in \mathrm{End}(A')^0$ such that the following diagram commutes in the isogeny category:*

$$\begin{array}{ccc} A & \xrightarrow[\pi]{} & A' \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T'} \\ A & \xrightarrow{\pi} & A' \end{array} \tag{2}$$

Then $T' \in \mathrm{End}(A')$.

Proof. We have $T_0 := nT' \in \mathrm{End}(A')$ for some nonzero integer n, and by the universal property of the quotient map $A' \to A'$ having kernel $A'[n]$, we have $T' \in \mathrm{End}(A')$ if and only if T_0 kills $A'[n]$. Since $T_0\pi = nT'\pi = n\pi T = \pi T n$ in the isogeny category, the genuine maps $T_0\pi$ and $\pi T n$ agree so that $T_0\pi$ kills $A'[n]$ as $\pi T n$ obviously does. To conclude that T_0 kills $A'[n]$, it therefore suffices to show that π is faithfully flat on n-torsion, for which it is enough to show surjectivity on $\overline{K}$-points (as finite K-groups are étale so $A[n] \xrightarrow{\pi} A'[n]$ is faithfully flat if and only if it is surjective on $\overline{K}$-points). But we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B(\overline{K}) & \longrightarrow & A(\overline{K}) & \longrightarrow & A'(\overline{K}) & \longrightarrow & 0 \\ & & {\scriptstyle n}\downarrow & & {\scriptstyle n}\downarrow & & {\scriptstyle n}\downarrow & & \\ 0 & \longrightarrow & B(\overline{K}) & \longrightarrow & A(\overline{K}) & \longrightarrow & A'(\overline{K}) & \longrightarrow & 0 \end{array}$$

and since A, A', B are abelian varieties and K is of characteristic 0, the vertical maps are all surjective, so by the Snake lemma, the map on n-torsion $A[n] \xrightarrow{\pi} A'[n]$ is surjective as desired. This completes the proof. ∎

Applying Theorem 4.3 to the optimal quotient $J_{\mathfrak{P}}$ of $J_0(N)$ shows that we have an action of $\mathbf{T}$ on $J_{\mathfrak{P}}$.

5. Admissibility

In Brian's talk, it was explained why $J_{\mathfrak{P}}[p]$ is an admissible group scheme over $\mathbf{Z}$. In this section we will recall the proof of this fact in more detail.

Since $\mathbf{T}$ is a finite $\mathbf{Z}$-module, the ring $\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p$ is a finite $\mathbf{Z}_p$-module and hence is semi-local. Moreover, we have seen in section 4 that $\mathbf{T}$ acts on $J_{\mathfrak{P}}$ in a manner that respects the quotient map $J_0(N)\to J_{\mathfrak{P}}$ so we obtain an action of $\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p$ on the Tate modules $T_p(J_0(N))$ and $T_p(J_{\mathfrak{P}})$ as $G_{\mathbf{Q}}:=\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-modules. By 8.7 and 8.15 of [3], there is a canonical isomorphism

$$\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p\simeq\prod_{\mathfrak{m}\in\mathrm{MaxSpec}(\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p)}(\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p)_{\mathfrak{m}}. \tag{3}$$

Claim 5.1. *The action of $\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p$ on $T_p(J_{\mathfrak{P}})$ factors through $\mathbf{T}_{\mathfrak{P}}:=(\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p)_{\mathfrak{P}}$, so the induced action of $\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p$ on $J_{\mathfrak{P}}[p]$ is through the quotient $\mathbf{T}_{\mathfrak{P}}$.*

Proof. First note that the functoriality of the idempotent decomposition of $\mathbf{T}\otimes\mathbf{Q}$ implies that

$$V_p(J_{\mathfrak{P}})=\prod_{\substack{\mathfrak{q}\in\mathrm{Spec}(\mathbf{T}\otimes\mathbf{Q})\\ \mathfrak{q}\subset\mathfrak{P}}}V_p(J_{\mathfrak{q}})=\prod_{\substack{\mathfrak{q}\in\mathrm{Spec}(\mathbf{T}\otimes\mathbf{Q})\\ \mathfrak{q}\subset\mathfrak{P}}}V_p(J)_{\mathfrak{q}},$$

where the objects $V_p(J)_{\mathfrak{q}}$ denote the localizations of the $\mathbf{T}\otimes\mathbf{Q}$-module $V_p(J)$ at primes $\mathfrak{q}$. (We will use the same letter $\mathfrak{q}$ to denote both a prime ideal of $\mathbf{T}\otimes\mathbf{Q}$ and its inverse image in $\mathbf{T}$). From this we see that the action of $\mathbf{T}$ on $V_p(J_{\mathfrak{P}})$ factors through $\prod_{\mathfrak{q}\subset\mathfrak{P}}(\mathbf{T}\otimes\mathbf{Q})_{\mathfrak{q}}$ and since every element of $\mathbf{T}-\mathfrak{P}$ is mapped to a unit of $\prod_{\mathfrak{q}\subset\mathfrak{P}}(\mathbf{T}\otimes\mathbf{Q})_{\mathfrak{q}}$ (since $\mathfrak{q}\subset\mathfrak{P}$), the map $\mathbf{T}\to\prod_{\mathfrak{q}\subset\mathfrak{P}}(\mathbf{T}\otimes\mathbf{Q})_{\mathfrak{q}}$ factors through the localization $\mathbf{T}_{\mathfrak{P}}$. The claim now follows after tensoring with $\mathbf{Z}_p$ and noting that $T_p(J_{\mathfrak{P}})\subseteq V_p(J_{\mathfrak{P}})$. ∎

We now prove:

Lemma 5.2. *Let $\mathscr{I}\subseteq\mathbf{T}$ be the Eisenstein ideal and $\mathfrak{P}\supset\mathscr{I}$ be any prime of $\mathbf{T}$ having residue characteristic $p\neq N$. Then $\mathfrak{P}^r$ kills $J_{\mathfrak{P}}[p]$ for some $r>0$.*

Proof. The action of $\mathbf{T}\otimes_{\mathbf{Z}}\mathbf{Z}_p$ on $J_{\mathfrak{P}}[p]$ factors through $\mathbf{T}_{\mathfrak{P}}/p\mathbf{T}_{\mathfrak{P}}$. As $\mathbf{T}\otimes\mathbf{Z}_p$ is a finite $\mathbf{Z}_p$-module, we see that $\mathbf{T}_{\mathfrak{P}}/p\mathbf{T}_{\mathfrak{P}}$ is a finite $\mathbf{F}_p$-module, and consequently is a finite local ring. It follows that its maximal ideal is nilpotent. Thus, for some $r>0$ we have $\mathfrak{P}^r\subseteq p\mathbf{T}_{\mathfrak{P}}$, so $\mathfrak{P}^rJ_{\mathfrak{P}}[p]=0$ as claimed. ∎

It follows from Lemma 5.2 that we have a filtration of $\mathbf{T}_{\mathfrak{P}}[G_{\mathbf{Q}}]$ modules

$$J_{\mathfrak{P}}[p]\supseteq\mathfrak{P}J_{\mathfrak{P}}[p]\supseteq\mathfrak{P}^2J_{\mathfrak{P}}[p]\supseteq\ldots\supseteq\mathfrak{P}^rJ_{\mathfrak{P}}[p]=0. \tag{4}$$

Observe that each quotient $\mathfrak{P}^iJ_{\mathfrak{P}}[p]/\mathfrak{P}^{i+1}J_{\mathfrak{P}}[p]$ is killed by $\mathfrak{P}$. We will now show that this implies that the quotients are all admissible. Recall from [2] (Corollary 1.6) that admissibility of a group scheme over $\mathbf{Z}$ can be checked on the corresponding Galois module of $\overline{\mathbf{Q}}$ points. It then follows that $J_{\mathfrak{P}}[p]$ is admissible, by general results on Jordan–Hölder series.

Lemma 5.3. *Let G be a finite discrete $G_{\mathbf{Q}}$-module of p-power order on which $\mathbf{T}$ acts and let $\mathfrak{P}\supseteq\mathscr{I}$ be a prime of $\mathbf{T}$ having residual characteristic $p\neq N$ and containing the Eisenstein ideal. Assume that for any prime $\ell\not|Np$ that the inertia group I_ℓ at ℓ acts trivially on G, and hence that we have an action of $\mathrm{Frob}_\ell\in\mathrm{Gal}(\overline{\mathbf{F}}_\ell/\mathbf{F}_\ell)$ on G. Suppose that the Eichler-Shimura relation*

$$\mathrm{Frob}_\ell^2-T_\ell\,\mathrm{Frob}_\ell+\ell=0$$

holds on G for all such ℓ. If $\mathfrak{P}$ kills G then G has a filtration by admissible closed subgroups with successive quotients $\mathbf{Z}/p\mathbf{Z}$ or μ_p.

Proof. Let Γ be the discrete finite quotient of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ through which the Galois action on G factors. By hypothesis, for $l\not|Np$ we have that Frob_ℓ acts on G and

$$\mathrm{Frob}_\ell^2-T_\ell\,\mathrm{Frob}_\ell+\ell=0 \tag{5}$$

on G. But as $\mathfrak{P}\supseteq\mathscr{I}$ kills G and $T_\ell\equiv\ell+1\bmod\mathscr{I}$, we see that Frob_ℓ acts on G with eigenvalues contained in $\{1,\ell\}$. Now any $\sigma\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ also acts on $G^\vee=\mathrm{Hom}(G,\mu_p)$ via $f^\sigma(g)=\sigma f(g^{\sigma^{-1}})$. Observe that if g is an eigenvector of $\sigma=\mathrm{Frob}_\ell$ with eigenvalue ℓ then $f^\sigma(g)=\sigma f(\ell^{-1}g)=\ell f(\ell^{-1}g)=f(g)$, so that $f(g)$ is an eigenvector with eigenvalue 1. Similarly, if g has eigenvalue 1 then $f(g)$ has eigenvalue ℓ. It follows that the eigenvalues $\{1, \ell\}$ of Frob_ℓ on $G \times G^\vee$ occur with the same multiplicity. Thus, the characteristic polynomial of Frob_ℓ on $G \times G^\vee$ over $\mathbf{F}_p$ is $(X-1)^d(X-\ell)^d$ for some d (which is the dimension of G as over $\mathbf{F}_p$).

Consider now the $\mathbf{F}_p[G_\mathbf{Q}]$-module $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$. By extending Γ if necessary (and still preserving the finiteness of Γ) we can regard $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$ as a finite discrete Γ-module. Since Frob_ℓ acts on μ_p with eigenvalue ℓ, we see that the characteristic polynomial of Frob_ℓ on $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$ is also $(X-\ell)^d(X-1)^d$. Now by the Tchebotarev density theorem, every $\gamma \in \Gamma$ is the image of Frob_ℓ for some $\ell \neq N, p$. Thus, every $\gamma \in \Gamma$ has the same characteristic polynomial on the two $\mathbf{F}_p[\Gamma]$-modules $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$ and G. Applying the Brauer–Nesbitt theorem, we conclude that these Γ-modules have the same semisimplifications. Since $(\mathbf{Z}/p\mathbf{Z})^d \times \mu_p^d$ has a filtration *as a Galois module* with successive quotients isomorphic to $\mathbf{Z}/p\mathbf{Z}$ or μ_p, so does G. ■

Corollary 5.4. *Let $N, p, \mathfrak{P}$ be as before. Then $J_\mathfrak{P}[p]$ is admissible.*

Proof. By [2] (Corollary 1.6), we need only check that the Galois module $J_\mathfrak{P}[p]$ has a $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-stable filtration by μ_p's and $\mathbf{Z}/p\mathbf{Z}$'s. Using Proposition 2.1 and Remark 2.2, we see that $J_\mathfrak{P}$ has toric reduction at N and good reduction outside N, so the inertia group at any $\ell \not| Np$ acts trivially on the Galois module $J_\mathfrak{P}[p](\overline{\mathbf{Q}})$. Moreover, the Eichler-Shimura relations hold on $J_\mathfrak{P}[p]$, as they do in $J_0(N)$ and hence on the isogeny factor $J_\mathfrak{P}$. We may therefore apply Lemma 5.3 to the situation discussed after the proof of Lemma 5.2 to conclude that $J_\mathfrak{P}[p]$ is admissible. ■

Corollary 5.5. *The nonzero isogeny factor $J_\mathfrak{P}$ of $J_0(N)$ has rank* 0.

Proof. By Proposition 2.1 and Remark 2.2, we know that $J_\mathfrak{P}$ has good reduction away from N and purely toric reduction at N. As $J_\mathfrak{P}[p]$ is admissible by Corollary 5.4, we see that $J_\mathfrak{P}$ satisfies the hypotheses of Theorem 1.1 and hence has rank 0. ■

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