BREUIL–KISIN MODULES VIA CRYSTALLINE COHOMOLOGY

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ABSTRACT. For a perfect field k of characteristic $p > 0$ and a smooth and proper formal scheme $\mathcal{X}$ over the ring of integers of a finite and totally ramified extension K of $W(k)[1/p]$, we propose a cohomological construction of the Breuil–Kisin modules attached to the p-adic étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$. We then prove that our proposal works when $p > 2$, $i < p - 1$, and the crystalline cohomology of the special fiber of $\mathcal{X}$ is torsion-free in degrees i and $i + 1$.

1. INTRODUCTION

Let k be a perfect field of characteristic $p > 0$ and K a finite and totally ramified extension of $W(k)[1/p]$. Fix an algebraic closure $\overline{K}$ of K and denote by $\mathbf{C}_K$ its p-adic completion. If $\mathcal{X}$ is a smooth proper formal scheme over $\mathcal{O}_K$ with (rigid analytic) generic fiber X, then the (torsion free part of the) p-adic étale cohomology $T^i := H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)/\text{tors}$ is a $G_K := \text{Gal}(\overline{K}/K)$-stable lattice in a crystalline representation. Functorially associated to the $\mathbf{Z}_p$-linear dual $(T^i)^\vee$ is its Breuil–Kisin module M^i over $\mathfrak{S} := W(k)[\![u]\!]$ in the sense of[1] [18]. It is natural to ask for a *direct cohomological* construction of M^i. While the work of Kisin [18, 2.2.7, A.6] provides a link between Dieudonné crystals and Breuil–Kisin modules for Barsotti–Tate representations, this link amounts to a descent result from Breuil modules over divided-power envelopes to Breuil–Kisin modules over $\mathfrak{S}$, which is somewhat indirect (and limited to the case of Hodge–Tate weights 0 and 1). More recently, the work of Bhatt, Morrow, and Scholze [4] associates to any smooth and proper formal scheme $\mathcal{X}$ over $\mathcal{O}_{\mathbf{C}_K}$ a perfect complex of A_{inf}-modules $R\Gamma_{A_{\text{inf}}}(\mathcal{X})$ whose cohomology groups are Breuil–Kisin–Fargues modules in the sense of [4, Def. 4.22] (see also Definition 4.4 below), and which is an avatar of all p-integral p-adic cohomology groups of $\mathcal{X}$. One can deduce from their theory that if $\mathscr{X} = \mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{\mathbf{C}_K}$ is defined over $\mathcal{O}_K$, then the base change $M^i_{A_{\text{inf}}} := M^i \otimes_{\mathfrak{S}} A_{\text{inf}}$ is a Breuil–Kisin–Fargues module, and one has a canonical identification $H^i(R\Gamma_{A_{\text{inf}}}(\mathscr{X})) \simeq M^i_{A_{\text{inf}}}$ under the assumption that $H^i_{\text{cris}}(\mathcal{X}_k/W(k))$ is torsion-free. We note that with this assumption, $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ is also torsion free; see Theorem 14.5 and Proposition 4.34 of [4].

Unfortunately, this beautiful cohomological description of $M^i \otimes_{\mathfrak{S}} A_{\text{inf}}$ does not yield a cohomological interpretation of the original Breuil–Kisin module M^i over $\mathfrak{S}$, but only of its scalar extension to A_{inf}, which is a coarser invariant.

[1]In fact, we work with a slightly different normalization than [18], which is more closely related to (crystalline) cohomology; see Definition 3.1 and Remark 3.2 for details.

In this paper, assuming that $H^j_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ is torsion-free for $j = i, i+1$, we will provide a direct, cohomological construction of M^i over $\mathfrak{S}$, at least when $i < p - 1$. To describe our construction, we must first introduce some notation.

Fix a uniformizer π_0 of $\mathscr{O}_K$, and let $E \in \mathfrak{S}$ be the minimal polynomial of π_0 over $W(k)$, normalized to have constant term p. For each $n \geq 1$ choose $\pi_n \in \mathscr{O}_{\overline{K}}$ satisfying $\pi_n^p = \pi_{n-1}$ and define $K_n := K(\pi_n)$ and $K_\infty := \cup_n K_n$. For $n \geq 0$ we define $\mathfrak{S}_n := W(k)[\![u_n]\!]$, equipped with the unique continuous Frobenius endomorphism φ that acts on $W(k)$ as the unique lift σ of the p-power map on k and satisfies $\varphi(u_n) = u_n^p$. We write $\theta_n : \mathfrak{S}_n \twoheadrightarrow \mathscr{O}_{K_n}$ for the continuous $W(k)$-algebra surjection carrying u_n to π_n, and we view $\mathfrak{S}_n$ as a subring of $\mathfrak{S}_{n+1}$ by identifying $u_n = \varphi(u_{n+1})$; this is compatible (via the θ_n) with the canonical inclusions $K_n \hookrightarrow K_{n+1}$. We then see that $\varphi : \mathfrak{S}_{n+1} \to \mathfrak{S}_n$ is a (σ-semilinear) isomorphism, so for $n \geq 1$ the element

$$(1.1) \qquad z_n := E\varphi^{-1}(E)\cdots\varphi^{1-n}(E) = E(u_0)E(u_1)E(u_2)\dots E(u_{n-1})$$

makes sense in $\mathfrak{S}_n$ and, as a polynomial in u_n, has zero-set the Galois-conjugates of $\pi_1, \pi_2, \dots, \pi_n$. Define $z_0 := 1$, so that $\varphi(z_n) = \varphi(E)z_{n-1}$ for $n \geq 1$.

Write S_n for the p-adic completion of the PD-envelope of θ_n, equipped with the p-adic topology. This is naturally a PD-thickening of $\mathscr{O}_{K_n}$, equipped with a descending filtration $\{\mathrm{Fil}^i S_n\}_{i\geq 0}$ obtained by taking the closure in S_n of the usual PD-filtration. The inclusions $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ uniquely extend to $S_n \hookrightarrow S_{n+1}$, and we henceforth consider S_n as a subring of S_{n+1} in this way. Note that φ uniquely extends to a continuous endomorphism $\varphi : S_n \to S_n$ which has image contained in S_{n-1} (see Lemma 2.1). We identify $u_0 = u$ and $\mathfrak{S}_0 = \mathfrak{S}$, and will frequently write $S := S_0$.

Given a smooth and proper formal scheme $\mathcal{X}$ over $\mathscr{O}_K$, we write $\mathcal{X}_n := \mathcal{X} \times_{\mathscr{O}_K} \mathscr{O}_{K_n}/(p)$ for the base change to $\mathscr{O}_{K_n}/(p)$. As $S_n \twoheadrightarrow \mathscr{O}_{K_n}/(p)$ is a divided power thickening, we can then form the crystalline cohomology $\mathscr{M}_n^i := H^i_{\mathrm{cris}}(\mathcal{X}_n/S_n)$ of $\mathcal{X}_n$ relative to S_n. It is naturally a finite-type S_n-module with a φ-semilinear endomorphism $\varphi_{\mathscr{M}} : \mathscr{M}_n^i \to \mathscr{M}_n^i$ and a descending and exhaustive filtration $\mathrm{Fil}^j \mathscr{M}_n^i$.

Give the localization $S_n[z_n^{-1}]$ the $\mathbf{Z}$-filtration by integral powers of z_n, and equip $\mathscr{M}_n^i[z_n^{-1}] = \mathscr{M}_n \otimes_{S_n} S_n[z_n^{-1}]$ with the tensor product filtration; that is, $\mathrm{Fil}^j(\mathscr{M}_n^i[z_n^{-1}]) := \sum_{a+b=j} z_n^a \mathrm{Fil}^b \mathscr{M}_n^i$, with the sum ranging over all integers a, b and taking place inside $\mathscr{M}_n^i[z_n^{-1}]$. We then define

$$\underline{M}^i(\mathcal{X}) = \varprojlim_{\varphi, n} \mathrm{Fil}^0(H^i_{\mathrm{cris}}(\mathcal{X}_n/S_n)[z_n^{-1}])$$

$$:= \{\{\xi_n\}_{n\geq 0} \ : \ \xi_n \in \mathrm{Fil}^0(\mathscr{M}_n^i \otimes_{S_n} S_n[z_n^{-1}]) \text{ and } (\varphi_{\mathscr{M}} \otimes \varphi)(\xi_{n+1}) = \xi_n \text{ for all } n \geq 0\}.$$

We equip $\underline{M}^i(\mathcal{X})$ with the Frobenius map $\varphi_{\underline{M}}(\{\xi_n\}) := \{(\varphi_{\mathscr{M}} \otimes \varphi)(\xi_n)\}_n$ and define

$$\mathrm{Fil}^j \underline{M}^i(\mathcal{X}) := \{\{\xi_n\}_n \in \underline{M}^i(\mathcal{X}) \ : \ \xi_0 \in \mathrm{Fil}^j(\mathscr{M}_0^i)\}.$$

We view $\underline{M}^i(\mathcal{X})$ as an $\mathfrak{S} = \mathfrak{S}_0$-module by $g(u) \cdot \{\xi_n\}_{n\geq 0} := \{g^{\sigma^{-n}}(u_n) \cdot \xi_n\}_{n\geq 0}$.

With these preliminaries, we can now state our main result, which provides a cohomological description of Breuil–Kisin modules in Hodge–Tate weights at most $p - 2$:

Theorem 1.1. *Assume that $p > 2$. Let $\mathcal{X}$ be a smooth and proper formal scheme over $\mathscr{O}_K$ and i an integer with $0 \leq i < p - 1$, and let M be the Breuil–Kisin module associated to the $\mathbf{Z}_p$-dual of the Galois lattice $H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Z}_p)/\mathrm{tors}$. If $H^j_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ is torsion-free for $j = i, i+1$, then there is a natural isomorphism of Breuil–Kisin modules*

$$M \simeq \underline{M}^i(\mathcal{X}).$$

The proof of Theorem 1.1 has two major—and fairly independent—ingredients, one of which might be described as purely cohomological, and the other as purely (semi)linear algebraic. Fix a

nonnegative integer $r < p - 1$, and let $\mathrm{Mod}_S^{\varphi,r}$ be the category of *height-r quasi-Breuil modules over* S, whose objects are triples $(\mathscr{M}, \mathrm{Fil}^r\,\mathscr{M}, \varphi_{\mathscr{M},r})$ where $\mathscr{M}$ is a finite, free S-module, $\mathrm{Fil}^r\,\mathscr{M} \subseteq \mathscr{M}$ is a submodule containing $(\mathrm{Fil}^r\,S)\mathscr{M}$ with the property that $\mathscr{M}/\mathrm{Fil}^r\,\mathscr{M}$ is p-torsion free, and $\varphi_{\mathscr{M},r} : \mathrm{Fil}^r\,\mathscr{M} \to \mathscr{M}$ is a φ-semilinear map whose image generates $\mathscr{M}$ as an S-module. Morphisms are filtration and φ-comaptible S-module homomorphisms. For each $j \in \mathbf{Z}$, we then define S-submodules $\mathrm{Fil}^j\,\mathscr{M} := \{m \in \mathscr{M} \; : \; E^{r-j}m \in \mathrm{Fil}^r\,\mathscr{M}\}$ for $j \leq r$ and we put $\mathrm{Fil}^j\,\mathscr{M} = 0$ for $j > r$. We similarly define the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ of *height-r filtered Breuil–Kisin modules*, whose objects are triples $(M, \mathrm{Fil}^r\,M, \varphi_{M,r})$ where M is a finite and free $\mathfrak{S}$-module, $\mathrm{Fil}^r\,M \subseteq M$ is a submodule containing $E^r M$ with $M/\mathrm{Fil}^r\,M$ having no p-torsion, and $\varphi_{M,r} : \mathrm{Fil}^r\,M \to M$ is a φ-semilinear map whose image generates M as an $\mathfrak{S}$-module, and we define $\mathrm{Fil}^j\,M := \{m \in M \; : \; E^{r-j}m \in \mathrm{Fil}^r\,M\}$ for $j \leq r$, with $\mathrm{Fil}^j\,M = 0$ when $j > r$. It is well-known that $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ is equivalent to the "usual" category of Breuil–Kisin modules over $\mathfrak{S}$; see Remark 3.2. Scalar extension along $\mathfrak{S} \to S$ induces a covariant functor $\underline{\mathscr{M}} : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Mod}_S^{\varphi,r}$ which is known to be an equivalence of categories [14, Theorem 2.2.1]. Our main "(semi)linear-algebraic" result is that the functor $\underline{M} : \mathrm{Mod}_S^{\varphi,r} \to \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ defined by $\underline{M}(\mathscr{M}) := \varprojlim_{\varphi,n} \mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}])$ is a quasi-inverse to $\underline{\mathscr{M}}$. This we establish using a structural result (Lemma 3.9) that provides an explicit description of a Breuil module via bases and matrices, together with a sequence of somewhat delicate Lemmas that rely on the fine properties of the rings S_n and their endomorphisms φ.

On the other hand, if $\mathscr{X}$ is a smooth and proper formal $\mathscr{O}_K$-scheme and $i \leq r < p - 1$, then the crystalline cohomology $\mathscr{M} := H^i_{\mathrm{cris}}(\mathscr{X}_0/S)$ can be naturally promoted to an object of $\mathrm{Mod}_S^{\varphi,i}$. Using the results of Bhatt, Morrow, and Scholze [4], when $H^j_{\mathrm{cris}}(\mathscr{X}_k/W(k))$ is torsion free for $j = i, i+1$, we prove in §5 that one has a canonical comparison isomorphism

$$\mathrm{Hom}_{S,\mathrm{Fil},\varphi}(\mathscr{M}, A_{\mathrm{cris}}) \simeq H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Z}_p),$$

from which we deduce that $\underline{M}(\mathscr{M})$ may be identified with the (filtered) Breuil–Kisin module M^i attached to the $\mathbf{Z}_p$-linear dual of $H^i_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbf{Z}_p)/\,\mathrm{tors}$. Theorem 1.1 then follows.

2. Ring-theoretic constructions

We keep the notation of §1. Note that, by the very definition, the ring S_n is topologically generated as an $\mathfrak{S}_n$-algebra by the divided powers $\{E^i/i!\}_{i \geq 1}$. It follows that $\mathrm{Fil}^i\,S_n$ is the closure of the expanded ideal $(\mathrm{Fil}^i\,S)S_n$ in S_n. We write $c_0 := \varphi(E)/p$, which is a unit of $S = S_0$. Since $\varphi(g) \equiv g^p \bmod p$, one shows that $c_0 = v + E^p/p$ for a unit $v \in \mathfrak{S}$. Observe that

$$\left(\frac{E^p}{p}\right)^n = \frac{(pn)!}{p^n}\left(\frac{E^{pn}}{(pn)!}\right), \text{ and } v_p((pn)!/p^n) = v_p(n!)$$

by Legendre's formula, so that the ring $T_n := \mathfrak{S}_n[\![E^p/p]\!]$ is naturally a subring of S_n that contains c_0 and is stable under φ as $\varphi(E) = pc_0$. There are obvious inclusions $T_n \hookrightarrow T_{n+1}$ that are compatible with the given inclusions $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ and $S_n \hookrightarrow S_{n+1}$. By definition, the injective map $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ has image precisely $\mathfrak{S}_{n-1}$ inside $\mathfrak{S}_n$. While the naïve analogue of this fact for the rings S_n is certainly *false*, Frobenius is nonetheless a "contraction" on S_n in the following precise sense:

Lemma 2.1. *Let i be a nonnegative integer and set $b(i) := \left\lceil i\frac{p-2}{p-1} \right\rceil$. We have $\varphi(S_n) \subseteq T_{n-1}$ inside S_n; in particular, $\varphi : S_n \to S_n$ has image contained in S_{n-1}. Moreover, if $x \in \mathrm{Fil}^i\,S_n$ then $\varphi(x) = w + y$ for some $w \in \mathfrak{S}_{n-1}$ and $y \in \mathrm{Fil}^{pb(i)}\,S_{n-1}$.*

Proof. Since $\mathrm{Fil}^i S_n$ is topologically generated as an $\mathfrak{S}_n$-module by $\{E^j/j!\}_{j \geq i}$ and $\varphi(\mathfrak{S}_n) = \mathfrak{S}_{n-1}$, to prove the first assertion it is enough to show that $\varphi(E^j/j!)$ lies in T_{n-1} for all j. But this is clear, as $\varphi(E^j/j!) = c_0^j p^j/j!$ and $p^j/j! \in \mathbf{Z}_p$ for all j. To prove the second assertion, it likewise suffices to treat only the casees $x = E^j/j!$ for $j \geq i$. As observed above, $\varphi(E) = E^p + pv$ for $v \in \mathfrak{S}_0^\times$, so we compute

$$(2.1) \qquad \varphi(E^j/j!) = \frac{(E^p + pz)^j}{j!} = \sum_{k=0}^{j} \frac{p^k}{(j-k)!k!} E^{p(j-k)} z^k.$$

Writing $s_p(n)$ for the sum of the p-adic digits of any nonnegative integer n and again invoking Legendre's formula gives

$$v_p\left(\frac{p^k}{(j-k)!k!}\right) = k - \frac{j}{p-1} + \frac{s_p(j-k) + s_p(k)}{p-1},$$

which is nonnegative for $k \geq j/(p-1)$. On the other hand, if $k < j/(p-1)$ then one has the inequality $(j-k) \geq \left\lceil j\frac{p-2}{p-1} \right\rceil = b(j)$. Combining these observations with (2.1) then gives the desired decomposition $\varphi(E^j/j!) = w + y$ with $w \in \mathfrak{S}_0$ the sum of all terms in (2.1) with $k \geq j/(p-1)$ and $y \in \mathrm{Fil}^{pb(j)} S_{n-1}$ the sum of the remaining terms. $\qquad \square$

We now define

$$\varprojlim_{\varphi, n} S_n := \{\{s_n\}_{n \geq 0} : s_n \in S_n \text{ and } \varphi(s_{n+1}) = s_n, \text{ for all } n \geq 0\},$$

which—as φ is a ring homomorphism—has the natural structure of a ring via component-wise addition and multiplication. The fact that φ "contracts" the tower of rings $\{S_n\}_{n \geq 0}$ manifests itself in the following Lemma, which inspired this paper:

Lemma 2.2. *The natural map*

$$(2.2) \qquad \mathfrak{S} \to \varprojlim_{\varphi, n} S_n, \qquad g(u) \mapsto \{g^{\sigma^{-n}}(u_n)\}_{n \geq 0}$$

is an isomorphism of rings.

Proof. It is clear that the given map is an injective ring homomorphism, so it suffices to prove that it is surjective. Let $\{s_n\}_{n \geq 0}$ be an arbitrary element of $\varprojlim_{\varphi, n} S_n$. Since $S_n = \mathfrak{S}_n + \mathrm{Fil}^p S_n$, an easy induction using Lemma 2.1 shows that $s_0 = \varphi^{(n)}(s_n)$ lies in $\mathfrak{S}_0 + \mathrm{Fil}^{i_n} S_0$, where i_n is defined recursively by $i_0 = p$ and $i_n = pb(i_{n-1})$ for $n \geq 1$. As this holds for all $n \geq 0$ and

$$i_{n+1} - i_n = pb(i_n) - i_n \geq pi_n\frac{p-2}{p-1} - i_n = (p-3)i_n + \frac{p-2}{p-1}i_n$$

so that $\{i_n\}_{n \geq 0}$ is an increasing sequence (recall $p > 2$), it follows that $s_0 \in \mathfrak{S}_0$. But then $\{s_n\}_{n \geq 0} = \{\varphi^{-n}(s_0)\}_{n \geq 0}$ is in the image of (2.2), as desired. $\qquad \square$

For later use, we record here the following elementary result:

Lemma 2.3. *Let n and m be any nonnegative integers. Then*

  (1) $\mathrm{Fil}^m S_n \cap \mathfrak{S}_n = E^m \mathfrak{S}_n$ *inside S_n.*
  (2) $(\mathrm{Fil}^m S_n)[1/p] \cap S_n = \mathrm{Fil}^m S_n$ *inside $S_n[1/p]$.*

Proof. We must prove that $\mathfrak{S}_n/E^m\mathfrak{S}_n \to S_n/\mathrm{Fil}^m S_n$ is injective with target that is p-torsion free. This is an easy induction on m, using the fact that $(E^m)/(E^{m+1})$ and $\mathrm{Fil}^m S_n/\mathrm{Fil}^{m+1} S_n$ are free of rank one over $\mathfrak{S}_n/(E) \simeq S_n/\mathrm{Fil}^1 S_n \simeq \mathcal{O}_{K_n}$ with generators E^m and $E^m/m!$, respectively. $\qquad \square$

3. Breuil and Breuil–Kisin modules

We begin by recalling the relation between Breuil–Kisin modules and Breuil modules in low Hodge–Tate weights. Throughout, we fix an integer $r < p - 1$.

Definition 3.1. We write $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ for the category of *height-r filtered Breuil–Kisin modules* over $\mathfrak{S}$ whose objects are triples $(M, \mathrm{Fil}^r M, \varphi_{M,r})$ where:

- M is a finite free $\mathfrak{S}$-module,
- $\mathrm{Fil}^r M$ is a submodule with $E^r M \subseteq \mathrm{Fil}^r M$ and $M/\mathrm{Fil}^r M$ is p-torsion free.
- $\varphi_{M,r} : \mathrm{Fil}^r M \to M$ is a φ-semilinear map whose image generates M as an $\mathfrak{S}$-module.

Morphisms are $\mathfrak{S}$-module homomorphisms which are compatible with the additional structures. For any object $(M, \mathrm{Fil}^r M, \varphi_{M,r})$ of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and any $i \leq r$, we put

$$(3.1) \qquad \mathrm{Fil}^i M := \left\{ m \in M \ : \ E^{r-i} m \in \mathrm{Fil}^r M \right\}$$

and set $\mathrm{Fil}^i M := 0$ for $i > r$, and define a φ-semilinear map $\varphi_M : M \to M$ by the condition

$$(3.2) \qquad \varphi_M(m) := \varphi_{M,r}(E^r m)$$

for $m \in M$. Note that for $m \in \mathrm{Fil}^r M$ we have $\varphi_M(m) = \varphi(E)^r \varphi_{M,r}(m)$.

Remark 3.2. Our definition of the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ is perhaps non-standard (*cf.* [14]). In the literature, one usually works instead with the category of *Breuil–Kisin modules* (without filtration), whose objects are pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ consisting of a finite free $\mathfrak{S}$-module $\mathfrak{M}$ and a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ whose linearization is killed by E^r, with evident morphisms. However, the assignment $(M, \mathrm{Fil}^r M, \varphi_{M,r}) \rightsquigarrow (\mathrm{Fil}^r M, E^r \varphi_{M,r})$ induces an equivalence between our category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and the "usual" category of Breuil–Kisin modules $(\mathfrak{M}, \varphi_{\mathfrak{M}})$. While this is fairly standard (e.g. [21, Lemma 8.2] or [27, Lemma 1]), for the convenience of the reader and for later reference, we describe a quasi-inverse.

Given $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ as above and writing $\widetilde{\varphi} : \varphi^* \mathfrak{M} \to \mathfrak{M}$ for the linearization of φ, there is a unique (necessarily injective) $\mathfrak{S}$-linear map

$$\psi : \mathfrak{M} \to \varphi^* \mathfrak{M} \quad \text{satisfying} \quad \widetilde{\varphi} \circ \psi = E^r \cdot \mathrm{id} \, .$$

The corresponding filtered Breuil–Kisin module over $\mathfrak{S}$ is then given by

$$(3.3) \qquad M := \varphi^* \mathfrak{M}, \ \mathrm{Fil}^r M := \psi(\mathfrak{M}) \text{ with } \varphi_{M,r}(\psi(m)) := 1 \otimes m.$$

Alternatively, as one checks easily, we have the description

$$(3.4) \qquad \mathrm{Fil}^r M = \mathrm{Fil}^r \varphi^* \mathfrak{M} = \{ x \in \varphi^* \mathfrak{M} \ : \ (1 \otimes \varphi)(x) \in E^r \mathfrak{M} \}.$$

From (3.3) it is clear that M and $\mathrm{Fil}^r M$ are then *free* $\mathfrak{S}$-modules, so that $M/\mathrm{Fil}^r M$ has projective dimension 1 over $\mathfrak{S}$. It follows from the Auslander–Buchsbaum formula and Rees' theorem that $M/\mathrm{Fil}^r M$ has depth 1 as an $\mathfrak{S}$-module, so since $\mathfrak{S}$ has maximal ideal $(u, p) = (u, E)$ and E is a zero-divisor on $M/\mathrm{Fil}^r M$, it must be that $\pi_0 = u \bmod E$ is *not* a zero divisor on $M/\mathrm{Fil}^r M$ and hence this quotient is p-torsion free and we really do get a filtered Breuil–Kisin module in this way.

We have chosen to work with our category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ of filtered Breuil–Kisin modules instead of the "usual" category of Breuil–Kisin modules as it is our category whose objects are inherently "cohomological", as we shall see.

Let $S = S_0$ be as above (*p*-adically completed PD-envelope of $\theta_0 : \mathfrak{S}_0 \twoheadrightarrow \mathscr{O}_K, u_0 \mapsto \pi_0$), and for $i \geq 1$ write $\mathrm{Fil}^i S \subseteq S$ for the (closure of the) ideal generated by $\{E^n/n!\}_{n \geq i}$. For $i \leq r$ One has $\varphi(\mathrm{Fil}^i S) \subseteq p^i S$, so we may define $\varphi_i : \mathrm{Fil}^i S \to S$ as $\varphi_i := p^{-i} \varphi$.

Definition 3.3. Denote by $\mathrm{Mod}_S^{\varphi,r}$ the category of *height-r* (*quasi*) *Breuil-modules* over S. These are triples $(\mathscr{M}, \mathrm{Fil}^r \mathscr{M}, \varphi_{\mathscr{M},r})$ consisting of a finite free S-module $\mathscr{M}$ with an S-submodule $\mathrm{Fil}^r \mathscr{M}$ and a φ-semilinear map $\varphi_{\mathscr{M},r} : \mathrm{Fil}^r \mathscr{M} \to \mathscr{M}$ such that:

(1) $(\mathrm{Fil}^r S)\mathscr{M} \subseteq \mathrm{Fil}^r \mathscr{M}$ and $\mathscr{M}/\mathrm{Fil}^r \mathscr{M}$ has no p-torsion.
(2) The image of $\varphi_{\mathscr{M},r}$ generates $\mathscr{M}$ as an S-module

Morphisms are S-module homomorphisms that are compatible with the additional structures. Given a quasi Breuil module $(\mathscr{M}, \mathrm{Fil}^r \mathscr{M}, \varphi_{\mathscr{M},r})$ of height r, for $i \leq r$ we set

$$(3.5) \qquad \mathrm{Fil}^i \mathscr{M} := \left\{ m \in \mathscr{M} \; : \; E^{r-i}m \in \mathrm{Fil}^r \mathscr{M} \right\}$$

and we put $\mathrm{Fil}^i \mathscr{M} := 0$ for $i > r$ and define $\varphi_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ by the recipe

$$\varphi_{\mathscr{M}}(m) := c_0^{-r} \varphi_{\mathscr{M},r}(E^r m) \quad \text{with} \quad c_0 = \varphi(E)/p \in S^\times.$$

Note that $\varphi_{\mathscr{M}} = p^r \varphi_{\mathscr{M},r}$ on $\mathrm{Fil}^r \mathscr{M}$; it follows that $\varphi_{\mathscr{M}}$ and $\varphi_{\mathscr{M},r}$ determine each other.

There is a canonical "base change" functor

$$(3.6) \qquad \underline{\mathscr{M}} : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Mod}_S^{\varphi,r}$$

defined as follows: if $(M, \mathrm{Fil}^r M, \varphi_{M,r})$ is an object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, then we define $\mathscr{M} := S \otimes_{\mathfrak{S}} M$ and $\varphi_{\mathscr{M}} := \varphi \otimes \varphi_M$, with $\mathrm{Fil}^r \mathscr{M}$ the submodule generated by the images of $S \otimes_{\mathfrak{S}} \mathrm{Fil}^r M$ and $\mathrm{Fil}^r S \otimes_{\mathfrak{S}} M$. Then by definition, the restriction of $\varphi_{\mathscr{M}}$ to $\mathrm{Fil}^r \mathscr{M}$ has image contained in $\varphi^r(E)\mathscr{M}$, so it makes sense to define $\varphi_{\mathscr{M},r} := p^{-r}\varphi_{\mathscr{M}}$ on $\mathrm{Fil}^r \mathscr{M}$. Using the definition of the category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, it is straightforward to check that this really does define a covariant functor from $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ to $\mathrm{Mod}_S^{\varphi,r}$.

Remark 3.4. Let $(M, \mathrm{Fil}^r M, \varphi_{M,r})$ be any filtered Breuil–Kisin module over $\mathfrak{S}$ with associated Breuil module $(\mathscr{M}, \mathrm{Fil}^r \mathscr{M}, \varphi_{\mathscr{M},r})$ over S. Writing $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ for the "classical" Breuil–Kisin module over $\mathfrak{S}$ given as in Remark 3.2 and $\varphi : \mathfrak{S} \to S$ for the composition of inclusion with Frobenius, one checks using (3.3) and (3.4) that we have $\mathscr{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with

$$\mathrm{Fil}^r \mathscr{M} = \{ m \in \mathscr{M} = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \; : \; (1 \otimes \varphi_{\mathfrak{M}})(m) \in \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \}$$

and $\varphi_{\mathscr{M},r}$ is the composite

$$\mathrm{Fil}^r \mathscr{M} \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} \mathrm{Fil}^r S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_r \otimes 1} S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \mathscr{M} \; .$$

It is known that the functor (3.6) is an equivalence of categories. When $r = 1$, this follows from work of Kisin [18, 2.2.7, A.6], albeit in an indirect way as the argument passes through Galois representations. Caruso and Liu [14] give a proof of this equivalence for general $r < p - 1$ by appealing to the work of Breuil and using pure (semi)linear algebra with bases and matrices. However, no existing proof provides what one could reasonably call a *direct* description of a quasi-inverse functor. We will use the ideas of section 5 to provide such a description. Before doing so, however, we work out an instructive example:

Example 3.5. The (filtered) Breuil–Kisin module attached to Tate module of the p-divisible group μ_{p^∞} is the object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,1}$ given by $M = \mathfrak{S} \cdot \mathfrak{e}$ on which Frobenius acts as $\varphi_M(\mathfrak{e}) = \varphi(E) \cdot \mathfrak{e}$, with $\mathrm{Fil}^1 M = M$ and $\varphi_{M,1}(\mathfrak{e}) = \mathfrak{e}$. The corresponding Breuil module $\mathscr{M} = S \cdot e$ is of rank 1 over S with Frobenius acting as $\varphi_{\mathscr{M}}(e) = \varphi(E) \cdot e$ and we have $\mathrm{Fil}^1 \mathscr{M} = \mathscr{M}$ with $\varphi_{\mathscr{M},1}(e) = c_0 e$ where $c_0 = \varphi(E)/p \in S^\times$. Defining $\lambda := \prod_{n \geq 0} \varphi^n(c_0)$, we have that $\lambda \in S^\times$ satisfies $\lambda/\varphi(\lambda) = c_0$. It follows that multiplication by λ carries $\mathscr{M}$ isomorphically onto the Breuil module given by the triple (S, S, φ).

Let $z_n \in \mathfrak{S}_n$ be as in (1.1) and give $S_n[z_n^{-1}]$ the $\mathbf{Z}$-filtration by powers of z_n. Let us define
$$\underline{M}(\mathscr{M}) := \{\{\xi_n\}_{n \geq 0} \; : \; \xi_n \in \mathrm{Fil}^0(\mathscr{M} \otimes_{S_0} S_n[z_n^{-1}]), \text{ and } (\varphi \otimes \varphi)(\xi_n) = \xi_{n-1}, \; n \geq 1\}$$
which we give the structure of an $\mathfrak{S} = \mathfrak{S}_0$-module by the rule
$$g(u_0) \cdot \{\xi_n\}_{n \geq 0} := \{g^{\sigma^{-n}}(u_n)\xi_n\}_{n \geq 0},$$
where each $\mathscr{M} \otimes_{S_0} S_n[z_n^{-1}]$ is viewed as a module over $\mathfrak{S}_n$ through the right factor and the canonical inclusion $\mathfrak{S}_n \hookrightarrow S_n$.

We then claim that the $\mathfrak{S}$-linear map
$$\iota : M = \mathfrak{S} \cdot \mathfrak{e} \to \underline{M}(\mathscr{M}) \qquad \text{determined by} \qquad \iota(\mathfrak{e}) := \{e \otimes z_n^{-1}\}_{n \geq 0}$$
is an isomorphism.

To see this, first note that the map is well defined as
$$e \otimes z_n^{-1} \in \mathrm{Fil}^1 \mathscr{M} \otimes \mathrm{Fil}^{-1} S_n[z_n^{-1}] \subseteq \mathrm{Fil}^0(\mathscr{M} \otimes S_n[z_n^{-1}])$$
and
$$(\varphi \otimes \varphi)(e \otimes z_n^{-1}) = \varphi(E)e \otimes (\varphi(E)z_{n-1})^{-1} = e \otimes z_{n-1}^{-1}$$
for all $n \geq 1$ (recall that $z_0 = 1$). It is clear from the very construction that ι is injective. To see surjectivity, we just observe that every element of $\xi_n \in \mathrm{Fil}^0(\mathscr{M} \otimes_{S_0} S_n[z_n^{-1}])$ may be written as a simple tensor $\xi_n = e \otimes s_n/z_n$ with $s_n \in S_n$. The condition that the ξ_n form a φ-compatible sequence is then simply that $\varphi(s_n) = s_{n-1}$, *i.e.* that $\{s_n\}_{n \geq 0}$ lies in the projective limit $\varprojlim_{\varphi,n} S_n$, which is exactly the image of $\mathfrak{S}_0$ under the natural map thanks to Lemma 2.2. It follows immediately from this that $\{\xi_n\}_{n \geq 0}$ lies in the image of ι, as desired.

Remark 3.6. The intrepid reader may wish to work out the analogue of this example for the Tate module of the p-divisible group $\mathbf{Q}_p/\mathbf{Z}_p$, whose associated filtered Breuil–Kisin module is given by $M = \mathfrak{S} \cdot \mathfrak{e}$ with $\mathrm{Fil}^1 M = EM$ and $\varphi_{M,1}(E \cdot \mathfrak{e}) = \mathfrak{e}$. The corresponding Breuil module is given by the triple $(S, \mathrm{Fil}^1 S, \varphi_1)$. As it turns out, this computation is significantly more involved, and requires Lemma 3.13 (for $d = 1$) to carry out successfully.

With this motivating example, we may now formulate our main result, which is an *explicit* description of a quasi-inverse to (3.6). This allows us to realize Breuil–Kisin modules with Hodge–Tate weights in $\{0, \ldots, p-2\}$ as "Frobenius-completed cohomology up the tower $\{K_n\}_n$."

Definition 3.7. For $(\mathscr{M}, \mathrm{Fil}^r \mathscr{M}, \varphi_{\mathscr{M},r})$ any object of $\mathrm{Mod}_S^{\varphi,r}$, we define
$$\underline{M}(\mathscr{M}) := \varprojlim_{\varphi,n} \mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}])$$
$$= \{\{\xi_n\}_{n \geq 0} \; : \; \xi_n \in \mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}]) \text{ and } (\varphi_{\mathscr{M}} \otimes \varphi)(\xi_n) = \xi_{n-1} \text{ for } n \geq 1\}$$
with filtration
$$\mathrm{Fil}^i \underline{M}(\mathscr{M}) := \{\{\xi_n\}_{n \geq 0} \in \underline{M}(\mathscr{M}) \; : \; \xi_0 \in \mathrm{Fil}^i(\mathscr{M}) \otimes_S S_0\}.$$
We equip $\underline{M}(\mathscr{M})$ with the Frobenius $\varphi_{\underline{M}}$ given by
$$\varphi_{\underline{M}}(\{\xi_n\}_{n \geq 0}) := \{(\varphi_{\mathscr{M}} \otimes \varphi)(\xi_n)\}_{n \geq 0}.$$
and give $\underline{M}(\mathscr{M})$ the structure of an $\mathfrak{S}$-module via
$$g \cdot \{\xi_n\}_{n \geq 0} := \{g^{\sigma^{-n}(u_n)\xi_n}\}_n \quad \text{for} \quad g \in \mathfrak{S} = \mathfrak{S}_0.$$
It is straightforward to check that $\varphi_{\underline{M}}$ is a φ-semilinear map on $\underline{M}(\mathscr{M})$.

We will see in Corollary 3.17 that the functor $\underline{M}$ so defined takes values in $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, so in particular the restriction of φ_M to $\mathrm{Fil}^r \underline{M}(\mathscr{M})$ is divisible by $\varphi(E)^r$ and $\varphi_{M,r} := \varphi(E)^{-r}\varphi_{\underline{M}}$ makes sense on $\mathrm{Fil}^r \underline{M}(\mathscr{M})$.

Theorem 3.8. *The construction $\mathscr{M} \rightsquigarrow \underline{M}(\mathscr{M})$ defines a covariant functor*

$$\underline{M} : \mathrm{Mod}_S^{\varphi,r} \to \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$$

that is moreover a quasi-inverse to the functor $\underline{\mathscr{M}}$ of (3.6).

We will establish Theorem 3.8 through a sequence of lemmas. We begin with a structural result for Breuil modules which shows, in particular, that the functor (3.6) is essentially surjective:

Lemma 3.9. *Let $\mathscr{M} \in \mathrm{Mod}_S^{\varphi,r}$. There is an S-basis $e_1, \ldots, e_d$ of $\mathscr{M}$ and matrices $A, B \in M_d(\mathfrak{S})$ such that:*

(1) *If $(\alpha_1, \ldots, \alpha_d) := (e_1, \ldots, e_d)A$ then*

$$\mathrm{Fil}^r \mathscr{M} = \bigoplus_{i=1}^{d} S\alpha_i + \mathrm{Fil}^p \mathscr{M}.$$

(2) *$c_0^{-r}\varphi_{\mathscr{M},1}(\alpha_i) = e_i$ for $1 \le i \le d$*
(3) *$(e_1, \ldots, e_d) \cdot E^r = (\alpha_1, \ldots, \alpha_d)B$*
(4) *$\varphi_{\mathscr{M}}(e_i) = e_i\varphi(B)$ for $1 \le i \le d$*
(5) *$AB = BA = E^r$.*

In particular, the $\mathfrak{S}$-module $M := \bigoplus_{i=1}^{d} \mathfrak{S}e_i$ with $\mathrm{Fil}^r M := \bigoplus_{i=1}^{d} \mathfrak{S}\alpha_i$ and $\varphi_{M,r}$ determined by $\varphi_{M,r}(\alpha_i) := e_i$ is an object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ whose image under (3.6) is $\mathscr{M}$.

Proof. This is [14, Lemma 2.2.2]. $\qquad\square$

Remark 3.10. We emphasize that the proof of Lemma 3.9 given in [14]—which relies on (the easy part of) [23, Lemma 4.1.1]—uses only (semi)linear algebra. While this result establishes the essential surjectivity of the functor (3.6), the proof that this functor is an equivalence given in [14, Theorem 2.2.1] relies on (a generalization of) the full-faithfulness result [19, 1.1.11], which uses certain auxilliary categories of torsion Breuil–Kisin and Breuil modules and a devissage argument to reduce to the p-torsion case, where the result is a consequence of (the proof of) [9, 3.3.2] using Lemma 2.1.2.1 and Proposition 2.1.2.2 of [8] and the argument of [7, Theorem 4.1.1]. In contrast, by writing down an explicit quasi-inverse to (3.6), our proof of Theorem 3.8 uses neither devissage nor any auxilliary categories, and in particular does not rely on [7], [8], [9], or [19].

In what follows, given an object $\mathscr{M}$ of $\mathrm{Mod}_S^{\varphi,r}$, an S-basis $e_1, \ldots, e_d$ of $\mathscr{M}$, and an S-algebra S', we will abuse notation slightly and again write $e_1, \ldots, e_d$ for the induced S'-basis of $\mathscr{M} \otimes_S S'$.

Lemma 3.11. *Let $\mathscr{M} \in \mathrm{Mod}_S^{\varphi,r}$, and let $\alpha_1, \ldots, \alpha_d$ be as in Lemma 3.9. For $n \ge 1$, any element ξ_n of $\mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}])$ may be expressed in the form*

$$\xi_n = z_n^{-r}(e_1, \ldots, e_d) \cdot (Ax_n + y_n)$$

with x_n a (column) vector in S_n^d and y_n a vector in $(\mathrm{Fil}^p S_n)^d$.

Proof. Assume $n \ge 1$ and observe first that for $0 \le i \le r$ we have the containment

$$\mathrm{Fil}^i \mathscr{M} \otimes_S \mathrm{Fil}^{-i}(S_n[z_n^{-1}]) \subseteq \mathrm{Fil}^r \mathscr{M} \otimes_S \mathrm{Fil}^{-r}(S_n[z_n^{-1}]).$$

Indeed, recalling that $z_n = E\varphi^{-1}(E) \cdots \varphi^{1-n}(E)$, we see that $(z_n/E) \in \mathfrak{S}_n$ and compute that any simple tensor on the left side has the form

$$m \otimes s z_n^{-i} = m \otimes s z_n^{r-i} z_n^{-r} = E^{r-i} m \otimes s(z_n/E)^{r-i} z_n^{-r}$$

with $m \in \mathrm{Fil}^i \mathscr{M}$, and this lies in $\mathrm{Fil}^r \mathscr{M} \otimes \mathrm{Fil}^{-r}(S_n[z_n^{-1}])$ thanks to the very definition (3.5) of Fil^i.

On the other hand, it follows immediately from Lemma 3.9 that any $\xi_n \in \mathrm{Fil}^r \mathscr{M} \otimes_S \mathrm{Fil}^{-r}(S_n[z_n^{-1}])$ may be written in the form

$$\xi_n = ((\alpha_1, \ldots, \alpha_d)x_n + (e_1, \ldots, e_d)y_n) \otimes z_n^{-r} = z_n^{-r}(e_1, \ldots, e_d) \cdot (Ax_n + y_n)$$

for vectors $x_n \in S_n^d$ and $y_n \in (\mathrm{Fil}^p S_n)^d$. $\qquad\square$

Lemma 3.12. *Assume $p > 2$ and let d and r be positive integers with $r < p - 1$. Let A be a $d \times d$ matrix with entries in $\mathfrak{S} = \mathfrak{S}_0$ such that there exists a $d \times d$ matrix B with entries in $\mathfrak{S}$ satisfying $BA = E^r I_d$. Let x_1 a vector in S_1^d, and assume that for all $n \geq 2$ there is a vector $x_n \in S_n^d$ with*

$$(3.7) \qquad\qquad \varphi(x_n) = Ax_{n-1}.$$

Then all coordinates of x_1 lie in $\mathfrak{S}_1$.

Proof. For ease of notation, if j is a positive integer, we will write $\mathrm{Fil}^j S_n^d$ for the submodule of S_n^d consisting of vectors all of whose components lie in $\mathrm{Fil}^j S_n$. Suppose given a sequence $\{x_n\}_{n \geq 1}$ as above. We will prove that for any $n > 1$, if x_n can be written as a sum $x_n = y_n + y_n'$ with $y_n \in \mathfrak{S}_n^d$ and $y_n' \in \mathrm{Fil}^j S_n^d$, then x_{n-1} can be written $x_{n-1} = y_{n-1} + y_{n-1}'$ where $y_{n-1} \in \mathfrak{S}_{n-1}^d$ and $y_{n-1}' \in \mathrm{Fil}^{pb(j)-r} S_{n-1}^d$. So assume $x_n = y_n + y_n'$ with y_n and y_n' as above. Applying Frobenius coordinate-wise and using Lemma 2.1 and our hypotheses, we find that

$$Ax_{n-1} = \varphi(x_n) = z_{n-1} + z_{n-1}'$$

with $z_{n-1} \in \mathfrak{S}_{n-1}^d$ and $z_{n-1}' \in \mathrm{Fil}^{pb(j)} S_{n-1}^d$. Multiplying both sides by B then gives

$$E^r x_{n-1} = B\varphi(x_n) = Bz_{n-1} + Bz_{n-1}' = w_{n-1} + w_{n-1}'$$

with $w_{n-1} \in \mathfrak{S}_{n-1}^d$ and $w_{n-1}' \in \mathrm{Fil}^{pb(j)} S_{n-1}^d$. Now $r < p - 1 \leq pb(j)$, from which it follows that $w_{n-1} = E^r y_{n-1}$ for some $y_{n-1} \in \mathfrak{S}_{n-1}^d$ thanks to Lemma 2.3. We may then write $w_{n-1}' = E^r y_{n-1}'$ with $y_{n-1}' \in \mathrm{Fil}^{pb(j)-r} S_{n-1}^d[1/p]$. But since $x_{n-1} = y_{n-1} + y_{n-1}'$ with x_{n-1} and y_{n-1} both having coordinates in S_{n-1}, we conclude again using Lemma 2.3 that y_{n-1}' has coordinates in $S_{n-1} \cap \mathrm{Fil}^{pb(j)-r} S_{n-1}[1/p] = \mathrm{Fil}^{pb(j)-r} S_{n-1}$ as desired.

To complete the proof, we observe that since $S_n = \mathfrak{S}_n + \mathrm{Fil}^p S_n$, it follows from repeated applications of the above fact that $x_1 = y_1 + y_1'$ with $y_1 \in \mathfrak{S}_1^d$ and y_1' in $\mathrm{Fil}^{j_n} S_1^d$, with j_n determined recursively by $j_1 = p$ and $j_n = pb(j_{n-1}) - r$ for $n > 1$. From the definition of $b(\cdot)$ in Lemma 2.1 and our hypothesis $r < p - 1$, we compute that for $n \geq 1$

$$j_{n+1} - j_n = pb(j_n) - r - j_n \geq (p-3)j_n + (p-2)\left(\frac{j_n}{p-1} - 1\right).$$

Using the hypothesis $p > 2$ and induction on n with base case $j_1 = p$, we deduce that $j_{n+1} > j_n$ for all $n > 0$, so that $\{j_n\}_{n \geq 0}$ is an *increasing* sequence of positive integers. Taking $n \to \infty$ then gives $x_1 \in \mathfrak{S}_1^d$ as desired. $\qquad\square$

Lemma 3.13. *In the situation of Lemma 3.12, let $x_1 \in S_1^d$ and $y_1 \in \mathrm{Fil}^p S_1^d$ and suppose that for all $n \geq 1$ there are vectors $x_{n+1} \in S_{n+1}^d$ and $y_{n+1} \in \mathrm{Fil}^p S_{n+1}^d$ such that*

$$(3.8) \qquad\qquad \varphi(Ax_{n+1} + y_{n+1}) = \varphi(A)(Ax_n + y_n).$$

Then there exists a vector $w \in \mathfrak{S}_1^d$ such that

$$Ax_n + y_n = A\varphi^{-1}(A)\ldots\varphi^{1-n}(A) \cdot \varphi^{1-n}(w).$$

for $n \geq 1$. In particular, $Ax_n + y_n$ has all coordinates in $\mathfrak{S}_n$.

Proof. Let $n \geq 1$. Since $S_n = \mathfrak{S}_n + \mathrm{Fil}^p S_n$, we may and do assume that x_n has all coordinates in $\mathfrak{S}_n$. Let us write T_n for the closure of the subring $\mathfrak{S}_n[E^p/p]$ inside S_n. We first claim that y_n has all coordinates in T_n. To see this, observe that as $y_{n+1} \in \mathrm{Fil}^p S_{n+1}^d$ by hypothesis, we may write $y_{n+1} = \sum_{j \geq p} w_j E^j/j!$ with w_j a vector in $\mathfrak{S}_{n+1}^d$ for all j. Using the recursion (3.8) to isolate y_n, we find

$$\varphi(A)y_n = \varphi(A)(\varphi(x_{n+1}) - Ax_n) + \sum_{j \geq p} \varphi(w_j)c_0^j \frac{p^j}{j!}.$$

Multiplying both sides by $\varphi(B)$ and dividing by p^r we find

$$c_0^r y_n = c_0^r(\varphi(x_{n+1}) - Ax_n) + \varphi(B)\sum_{j \geq p} \varphi(w_j)c_0^j \frac{p^{j-r}}{j!},$$

and a standard calculation shows that for $j \geq p$ and $r \leq p-1$ we have $v_p(p^{j-r}/j!) \geq 0$. As the right side then clearly has coordinates in T_n, our claim follows.

Now we may write $y_n = \sum_{i \geq 0} w_i(E^p/p)^i$ with $w_i \in \mathfrak{S}_n^d$ for all i. Since y_n has coordinates in $\mathrm{Fil}^p S_n$, we must have $w_0 = E^p v_0$ for some $v_0 \in \mathfrak{S}_n^d$ and we compute that

$$py_n = E^p pv_0 + E^p \sum_{i \geq 1} w_i \left(\frac{E^p}{p}\right)^{i-1}.$$

In particular, $py_n = E^p y_n' = A(BE^{p-r}y_n') = At_n$ for some t_n with coordinates in T_n. Then $p(Ax_n + y_n) = A(px_n + t_n) = As_n$ with s_n a vector with all coordinates in $T_n \subseteq S_n$. Multiplying (3.8) by p and replacing $p(Ax_n + y_n)$ by $As_\star$ gives the recurrence

$$\varphi(s_n) = As_{n-1},$$

for all $n > 1$, which forces $s_1 \in \mathfrak{S}_1^d$ thanks to Lemma 3.12. For $n \geq 1$ we then have

$$(3.9) \qquad p(Ax_n + y_n) = As_n = A\varphi^{-1}(A)\ldots\varphi^{1-n}(A)\varphi^{1-n}(s_1).$$

To complete the proof, it therefore suffices to prove that s_1 has all coordinates divisible by p in $\mathfrak{S}_1$.

Multiplying 3.9 through by $C := B\varphi^{-1}(B)\ldots\varphi^{1-n}(B)$ gives

$$(3.10) \qquad pC(Ax_n + y_n) = E^r\varphi^{-1}(E^r)\ldots\varphi^{1-n}(E^r)\varphi^{1-n}(s_1) = z_n^r \varphi^{1-n}(s_1),$$

Since $pC(Ax_n + y_n)$ has coordinates in pS_n, we certainly have that all coordinates of $z_n^r\varphi^{1-n}(s_1)$ are zero in S_n/pS_n. On the other hand, from the very definition of S_n, we have an injection $k[u_n]/(u_n^{ep^{n+1}}) \hookrightarrow S_n/pS_n$, where e is the u_0-degree of $E = E(u_0)$. Let us write $s_1 = (s_{11}(u_1), \ldots, s_{1d}(u_1))$ with $s_{1j} \in \mathfrak{S}_1$, so that $\varphi^{1-n}(s_1)$ has coordinates $s_{1j}^{\sigma^{1-n}}(u_n) \in \mathfrak{S}_n$ for $1 \leq j \leq d$. Since

$$z_n \equiv u_0^e u_1^e \ldots u_{n-1}^e \equiv u_n^{pe\frac{p^n-1}{p-1}} \bmod p,$$

it follows from the above that the reduction modulo p of each coordinate $s_{1j}^{\sigma^{1-n}}(u_n) \in \mathfrak{S}_n$ is divisible by $u_n^{ei_n}$ in $k[\![u_n]\!]$ for *all* $n \geq 1$, where

$$i_n = p\left(p^n - r\frac{p^n-1}{p-1}\right).$$

This implies that $s_{1j}(u_1)$ mod p is divisible by $u_1^{ei_n}$ for *all* $n \geq 1$ and j. Again a straightforward calculation using the hypothesis $r < p-1$ shows that $i_n \to \infty$ as $n \to \infty$, and we conclude that $s_{1j}(u_1) \equiv 0$ mod p for all j, whence s_1 has all coordinates divisible by p in $\mathfrak{S}_1$, as desired. $\qquad\square$

Let $(M, \mathrm{Fil}\,M, \varphi_1)$ be an arbitrary object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and let $\varphi_M : M \to M$ be as in (3.2). Give the ring $\mathfrak{S}_n[z_n^{-1}]$ the $\mathbf{Z}$-filtration by powers of z_n, and for ease of notation, set

$$M_n := \mathrm{Fil}^0(M \otimes_{\mathfrak{S}} \mathfrak{S}_n[z_n^{-1}]).$$

Lemma 3.14. *For $n \geq 0$ and $x \in M_n$, there exists $y \in M_{n+1}$ with $(\varphi_M \otimes \varphi)(y) = x$. Moreover, y is unique.*

Proof. Since the image of $\varphi_{M,r} : \mathrm{Fil}^r M \to M$ generates M as an $\mathfrak{S}$-module, every element of M_n is a sum of elements of the form $\varphi_{M,r}(m) \otimes (s/z_n^r)$, for appropriate $m \in \mathrm{Fil}^r M$ and $s \in \mathfrak{S}_n$. Consider the element $m \otimes (\varphi^{-1}(s)/z_{n+1}^r)$, which lies in $M_{n+1} = \mathrm{Fil}^0(M \otimes_{\mathfrak{S}} \mathfrak{S}_{n+1}[z_{n+1}^{-1}])$. Then:

$$\begin{aligned}
(\varphi_M \otimes \varphi)(m \otimes (\varphi^{-1}(s)/z_{n+1}^r)) &= (\varphi_M(m)) \otimes (s/\varphi(z_{n+1}^r)) \\
&= \varphi(E)^r \varphi_{M,r}(m) \otimes (s/\varphi(E)^r z_n^r) \\
&= \varphi_{M,r}(m) \otimes (s/z_n^r)
\end{aligned}$$

This proves the existence of y as in the statement of the lemma. Uniqueness follows immediately from the fact that $\varphi_M \otimes \varphi$, viewed as a self-map of $M \otimes_{\mathfrak{S}} \mathfrak{S}_{n+1}[z_{n+1}^{-1}]$, is injective. $\qquad\square$

Remark 3.15. The Lemma shows the stronger fact that any $x \in M \otimes_{\mathfrak{S}} \mathrm{Fil}^{-r}(\mathfrak{S}_n[z_n^{-1}])$ has a unique preimage under $\varphi_M \otimes \varphi$ in $\mathrm{Fil}^r M \otimes \mathrm{Fil}^{-r}(\mathfrak{S}_n[z_{n+1}^{-1}])$.

Now let $(\mathscr{M}, \mathrm{Fil}^r \mathscr{M}, \varphi_{\mathscr{M},r}) := \underline{\mathscr{M}}(M)$ be the functorially associated object of $\mathrm{Mod}_S^{\varphi,r}$, so $\mathscr{M} = M \otimes_{\mathfrak{S}} S$ and $\mathrm{Fil}^r \mathscr{M}$ is S-submodule of $\mathscr{M}$ generated by the images of $M \otimes_{\mathfrak{S}} \mathrm{Fil}^r S$ and $\mathrm{Fil}^r M \otimes_{\mathfrak{S}} S$ under the obvious maps. As such, we have a canonical inclusion of $\mathfrak{S}_n$-modules:

$$\iota_n : M_n := \mathrm{Fil}^0(M \otimes_{\mathfrak{S}} \mathfrak{S}_n[z_n^{-1}]) \longrightarrow \mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}])$$

that is φ-compatible. We also have an obvious *isomorphism* $\tau : M \to M_0$ given by $m \mapsto m \otimes 1$. Given $m \in M$, for $n \geq 0$ we then define $\xi_n \in M_n$ to be the unique element of M_n satisfying

$$(\varphi_M \otimes \varphi)^{(n)}(\xi_n) = \tau(m);$$

this exists thanks to Lemma 3.14. We obtain a map:

(3.11) $$M \to \underline{M}(\mathscr{M}) = \varprojlim_{\varphi,n} \mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}]) \qquad \text{given by} \qquad m \mapsto \{\iota_n(\xi_n)\}_{n \geq 0},$$

Lemma 3.16. *The map* (3.11) *is a natural isomorphism of filtered φ-modules over $\mathfrak{S}$.*

Proof. We first prove that (3.11) is an isomorphism at the level of $\mathfrak{S}$-modules. Suppose that $\{\xi_n\}_{n \geq 0}$ is an arbitrary element of $\varprojlim_{\varphi,n}(\mathscr{M} \otimes_S S_n[z_n^{-1}])$. It suffices to prove that ξ_0 lies in the image of the canonical inclusion

$$\iota_0 \circ \tau : M \xrightarrow{\;\simeq\;}_{\tau} M_0 = \mathrm{Fil}^0(M \otimes_{\mathfrak{S}} \mathfrak{S}_0) \xrightarrow{\;\;}_{\iota_0} \mathrm{Fil}^0(\mathscr{M} \otimes_S S_0) \,.$$

Indeed, then projection $\{\xi_n\}_{n \geq 0} \mapsto \xi_0$ followed by the inverse of $\iota_0 \circ \tau$ on its image provides the desired inverse map to (3.11).

To do this, we identify M with its image under $\iota_0 \circ \tau$ and compute with bases. The map $\varphi_{M,r} : \varphi^* \mathrm{Fil}^r M \to M$ is a linear isomorphism of $\mathfrak{S}$-modules, so since $\varphi : \mathfrak{S} \to \mathfrak{S}$ is faithfully flat, $\mathrm{Fil}^r M$ is finite and free over $\mathfrak{S}$ with rank equal to that of M; this fact also follows easily from the

discussion of Remark 3.2. Fix an $\mathfrak{S}$-basis $\alpha_1, \ldots, \alpha_d$ for $\mathrm{Fil}^r M$ and set $e_i := \varphi_{M,r}(\alpha_i)$, so that e_i is then an $\mathfrak{S}$-basis of M. Since $E^r M \subseteq \mathrm{Fil}^r M$, we obtain matrices $A, B \in M_d(\mathfrak{S})$ determined by the conditions

$$(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots e_d) A \quad \text{and} \quad (e_1, \ldots, e_d) E^r = (\alpha_1, \ldots, \alpha_d) B$$

so that $AB = BA = E^r$. Note that the associated Breuil module $\mathscr{M}$ admits the "explicit" description as in Lemma 3.9.

Thanks to Lemma 3.11, for all $n \geq 1$ we may write

$$\xi_n = z_n^{-r}(e_1, \ldots, e_d) \cdot (A x_n + y_n)$$

for vectors $x_n \in S_n^d$ and $y_n \in \mathrm{Fil}^p S_n^d$. For $n \geq 1$ we then compute

$$\xi_n = (\varphi_{\mathscr{M}} \otimes \varphi)(\xi_{n+1}) = \varphi(E)^{-r} z_n^{-r}(e_1, \ldots, e_d)\varphi(B)\varphi(A x_{n+1} + y_{n+1})$$

so multiplying both sides by $z_n^r \varphi(E)^r$ and using the definition of ξ_n gives

$$\varphi(E)^r(A x_n + y_n) = \varphi(B)\varphi(A x_{n+1} + y_{n+1})$$

as (column) vectors in S_n^d, because $\{e_i\}$ is an S_n-basis of $\mathscr{M} \otimes_S S_n$. Multiplying this equality through by $\varphi(A)$, and cancelling the resulting factor of $\varphi(E)^r = \varphi(A)\varphi(B)$ from both sides finally yields the recurrence

$$\varphi(A x_{n+1} + y_{n+1}) = \varphi(A)(A x_n + y_n).$$

for $n \geq 1$. But now we are in precisely the situation of Lemma 3.13, which guarantees that $A x_1 + y_1 = A w_1$ for some $w_1 \in \mathfrak{S}^d$ so that

$$\xi_0 = (\varphi_{\mathscr{M}} \otimes \varphi)(\xi_1) = \varphi(z_1)^{-r}(e_1, \ldots, e_d)\varphi(B)\varphi(A)\varphi(w_1) = (e_1, \ldots, e_d)\varphi(w_1)$$

lies in M, as desired.

That the map (3.11) is Frobenius-compatible and carries $\mathrm{Fil}^r M$ into $\mathrm{Fil}^r \underline{M}(\mathscr{M})$ is clear from definitions. To check that it induces an isomorphism on filtrations, it suffices to prove that projection $\{\xi_n\}_{n \geq 0} \mapsto \xi_0$ is filtration-compatible. This amounts to the assertion that $\mathrm{Fil}^r \mathscr{M} \cap M \subseteq \mathrm{Fil}^r M$ inside $\mathscr{M}$. To verify this, as before, we may write any element of $\mathrm{Fil}^r \mathscr{M}$ as $(e_1, \ldots, e_d)(A x + y)$ with $x \in \mathfrak{S}^d$ and $y \in \mathrm{Fil}^p S^d$. If this is equal to some element $(e_1, \ldots, e_d)w$ of M with $w \in \mathfrak{S}^d$, then we must have $A x + y = w$ in S^d. Myltiplying both sides by B gives $E^r x + B y = B w$ so since $B y \in \mathrm{Fil}^p S^d$ we deduce that the coordinates of $B w$ lie in $\mathrm{Fil}^r S \cap \mathfrak{S} = E^r \mathfrak{S}$ thanks to Lemma 2.3. Then since $x \in \mathfrak{S}^d$, it follows that $B y$ has coordinates in $\mathrm{Fil}^p S \cap \mathfrak{S} = E^p \mathfrak{S}$, and we may write $B w = E^r v = B A v$ for some $v \in \mathfrak{S}^d$. This implies that $w = A v$ and hence that $(e_1, \ldots, e_d)w = (\alpha_1, \ldots, \alpha_d)v$ lies in $\mathrm{Fil}^r M$ as desired. $\square$

Corollary 3.17. *Let $\mathscr{M}$ be any object of $\mathrm{Mod}_S^{\varphi,r}$. Then $\underline{M}(\mathscr{M})$ is an object of $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$.*

Proof. This follows immediately from Lemmas 3.9 and 3.16. $\square$

We now have functors $\underline{\mathscr{M}} : \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r} \to \mathrm{Mod}_S^{\varphi,r}$ and $\underline{M} : \mathrm{Mod}_S^{\varphi,r} \to \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ and a functorial isomorphism $\underline{M} \circ \underline{\mathscr{M}} \simeq \mathrm{id}$ on $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$. To complete the proof of Theorem 3.8, it therefore remains to exhibit a natural transformation $\underline{\mathscr{M}} \circ \underline{M} \simeq \mathrm{id}$ of functors on $\mathrm{Mod}_S^{\varphi,r}$.

Let $(\mathscr{M}, \mathrm{Fil}^r \mathscr{M}, \varphi_{\mathscr{M},r})$ be any object of $\mathrm{Mod}_S^{\varphi,r}$. We define an S-linear map

$$(3.12) \quad \underline{\mathscr{M}}(\underline{M}(\mathscr{M})) = S \otimes_{\mathfrak{S}} \varprojlim_{\varphi,n} \mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}]) \to \mathscr{M} \otimes_S S_0 \simeq \mathscr{M} \quad \text{by} \quad s \otimes \{\xi_n\} \mapsto s \cdot \xi_0.$$

Lemma 3.18. *The map (3.12) is a natural isomorphism of filtered φ-modules over S.*

Proof. Naturality in $\mathscr{M}$ is clear, as is compatibility with Frobenius. By the very definition (3.6) of $\underline{\mathscr{M}}$, the submodule $\mathrm{Fil}^r(\underline{\mathscr{M}}(\underline{M}(\mathscr{M})))$ is generated by the images in $S \otimes_{\mathfrak{S}} \underline{M}(\mathscr{M})$ of $S \otimes_{\mathfrak{S}} \mathrm{Fil}^r \underline{M}(\mathscr{M})$ and $\mathrm{Fil}^r S \otimes_{\mathfrak{S}} \underline{M}(\mathscr{M})$, so due to Definition 3.7, any element of this submodule is a sum of simple tensors $s \otimes \{\xi_n\}$ with either $s \in \mathrm{Fil}^r S$ or $\xi_0 \in \mathrm{Fil}^r \mathscr{M}$. Since $(\mathrm{Fil}^r S)\mathscr{M} \subseteq \mathrm{Fil}^r \mathscr{M}$, it follows at once that the map (3.12) is compatible with filtrations.

Let us prove that (3.12) is an isomorphism. Thanks to Corollary 3.17, the map (3.12) is an S-linear map of *free* S-modules of the same rank, so it suffices to prove that it is surjective. Let $(e_1, \ldots, e_d)$, $(\alpha_1, \ldots, \alpha_d)$, A and B be as in Lemma 3.9. It is clearly enough to prove that e_i is in the image of (3.12) for each i. For $n \geq 1$ we define

$$\xi_{i,n} := z_n^{-r}(e_i A)\varphi^{-1}(A) \cdots \varphi^{1-n}(A).$$

As $\alpha_i = e_i A$ lies in $\mathrm{Fil}^r \mathscr{M}$, this really is an element of $\mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}])$ for $n \geq 1$ and we set $\xi_{i,0} := e_i$. We then compute for $n \geq 1$

$$(\varphi_{\mathscr{M}} \otimes \varphi)(\xi_{i,n}) = \varphi(E)^{-r} z_{n-1}^{-r} e_i \varphi(B)\varphi(A)A \cdots \varphi^{2-n}(A)$$
$$= z_{n-1}^{-r} e_i A \varphi^{-1}(A) \cdots \varphi^{2-n}(A) = \xi_{i,n-1}$$

so that $\xi_i := \{\xi_{i,n}\}_{n \geq 0}$ lies in $\underline{M}(\mathscr{M})$ and $1 \otimes \xi_i$ maps to e_i via (3.12).

Finally, we must check that the map on Fil^r's is an isomorphism, and to do so it suffices to prove that it is surjective. We know from Lemma 3.9 that any element $m \in \mathrm{Fil}^r \mathscr{M}$ may be expressed as $m = (e_1, \ldots e_d)(Ax + y)$ where $x \in \mathfrak{S}^d$ and $y \in \mathrm{Fil}^r S^d$. For $n \geq 1$ define

$$\nu_n := z_n^{-r}(e_1, \ldots, e_d)A\varphi^{-1}(A) \cdots \varphi^{1-n}(A)\varphi^{-n}(A)\varphi^{-n}(x),$$

which again lies in $\mathrm{Fil}^0(\mathscr{M} \otimes_S S_n[z_n^{-1}])$, and put $\nu_0 := (e_1, \ldots, e_d)Ax$, which lies in $\mathrm{Fil}^r \mathscr{M}$. Then as before one checks that $\nu := \{\nu_n\}$ is an element of $\mathrm{Fil}^r \underline{M}(\mathscr{M})$ with $1 \otimes \nu$ mapping to ν_0. Since $y = (y_1, \ldots, y_d)^T \in \mathrm{Fil}^r S^d$, the element $\eta := \sum_i y_i \otimes \xi_i$ lies in $\mathrm{Fil}^r S \otimes \underline{M}(\mathscr{M})$ and maps to $(e_1, \ldots, e_d)y$ under (3.12). Thus, the sum $1 \otimes \nu + \eta$ is an element of $\mathrm{Fil}^r \mathscr{M}(\underline{M}(\mathscr{M}))$ mapping to m, and the map on filtrations is surjective, as desired. $\square$

4. LATTICES IN GALOIS REPRESENTATIONS

In this section, we briefly review the relationship between the semilinear algebra categories of §3 and (stable lattices in) Galois representations.

We keep the notation of §1, and begin by recalling the definitions of the period rings that we will need. Let $R := \varprojlim \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}$, with the projective limit taken along the map $x \mapsto x^p$. Then R is a perfect valuation ring of equicharacteristic p and residue field $\overline{k}$, equipped with a natural coordinate-wise action of G_K. We put $A_{\mathrm{inf}} := W(R)$, and denote by $\theta_{\mathrm{inf}} : A_{\mathrm{inf}} \to \mathscr{O}_{\mathbf{C}_K}$ the unique ring homomorphism lifting the projection $R \to \mathscr{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of A_{inf} with respect to the ideal $\ker(\theta_{\mathrm{inf}})$. As usual, we write $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ and define B_{dR}^+ to be the $\ker(\theta_{\mathrm{inf}}[1/p])$-adic completion of $A_{\mathrm{inf}}[1/p]$. For any subring $A \subset B_{\mathrm{dR}}^+$, we define $\mathrm{Fil}^i A = A \cap (\ker(\theta_{\mathrm{dR}}))^i B_{\mathrm{dR}}^+$, with $\theta_{\mathrm{dR}} : B_{\mathrm{dR}}^+ \twoheadrightarrow \mathbf{C}_K$ the map induced by θ_{inf}.

Recall that we have fixed a compatible seqience $\{\pi_i\}_{i \geq 0}$ of p-power roots of our fixed uniformizer $\pi_0 \in K$. Then $\{\pi_i\}_{i \geq n}$ defines an element $\underline{\pi}_n \in R$, and we write $[\underline{\pi}_n] \in A_{\mathrm{inf}}$ for its Techmüller lift. For each n, we then embed the $W(k)$-algebra $W(k)[u_n]$ into $A_{\mathrm{inf}} \subset A_{\mathrm{cris}}$ by the map $u_n \mapsto [\underline{\pi}_n]$. These maps extend to embeddings $\mathfrak{S}_n \hookrightarrow A_{\mathrm{inf}}$ which intertwine the given Frobenius endomorphism on $\mathfrak{S}_n$ with the Witt vector Frobenius on A_{inf}, and which are compatible with the $W(k)$-algebra inclusions $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ that identify $\varphi(u_{n+1}) = u_n$. As before, we omit the subscript when it is zero, and simply write $\mathfrak{S} = \mathfrak{S}_0$ and $u = u_0$.

Recall that S_n is the p-adic completion of the divided power envelope of $\mathfrak{S}_n$ with respect to the ideal generated by $E(u_0) = E_n(u_n)$, equipped with the p-adic topology. We write $\mathrm{Fil}^m S \subset S$ for the closure of the ideal generated by $\gamma_i(E(u)) := \frac{E_n(u_n)^i}{i!}$ with $i \geq m$. Using the fact that $\ker(\theta_{\inf})$ is principally generated by $E([\underline{\pi}_0]) = E_n([\underline{\pi}_n])$, it is not difficult to prove that the embeddings $\mathfrak{S}_n \hookrightarrow A_{\inf}$ uniquely extend to continuous $W(k)$-algebra embeddings $S_n \hookrightarrow A_{\mathrm{cris}}$ compatible with Frobenius φ and filtration. As in §1, we write $K_n := K(\pi_n)$ and set $K_\infty := \cup_n K_n$. We define $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$ and note that we in fact have $\mathfrak{S}_n \subset A_{\inf}^{G_\infty}$ and $S_n \subset A_{\mathrm{cris}}^{G_\infty}$.

With these preliminaries, we now define certain functors from the categories of (filtered) Breuil–Kisin and Breuil modules to the category of Galois representations on finite free $\mathbf{Z}_p$-modules.

Let $M \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ be a filtered Breuil–Kisin module. Remembering that $\mathrm{Fil}^r A_{\inf} = E(u)^r A_{\inf}$, we define $\varphi_{A_{\inf},r} : \mathrm{Fil}^r A_{\inf} \to A_{\inf}$ by $\varphi_{A_{\inf},r}(E(u)^r x) := \varphi(x)$, and set

(4.1) $\underline{T}_{\mathfrak{S}}(M) := \mathrm{Hom}_{\mathfrak{S},\mathrm{Fil}^r,\varphi_r}(M, A_{\inf})$,

which, as one checks easily, is naturally a $\mathbf{Z}_p[G_\infty]$-module. Similarly, the restriction of φ on A_{cris} to $\mathrm{Fil}^r A_{\mathrm{cris}}$ has image in $p^r A_{\mathrm{cris}}$, so we may define $\varphi_{A_{\mathrm{cris}},r} : \mathrm{Fil}^r A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ by $\varphi_{A_{\mathrm{cris}},r} = \varphi_{A_{\mathrm{cris}}}/p^r$. For any quasi-Breuil module $\mathscr{M} \in \mathrm{Mod}_S^{\varphi,r}$ we may then attach the $\mathbf{Z}_p[G_\infty]$-module

(4.2) $\underline{T}_{\mathrm{cris}}(\mathscr{M}) := \mathrm{Hom}_{S,\mathrm{Fil}^r,\varphi_r}(\mathscr{M}, A_{\mathrm{cris}})$.

Before proceeding further, we recall a variant of the functor $\underline{T}_{\mathfrak{S}}$ on the category of (classical) Breuil–Kisin modules of Remark 3.2. Let $M \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, and let $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ be the associated classical Breuil–Kisin module. Then, as in (3.3) and (3.4), we have $M = \varphi^* \mathfrak{M}$ with $\varphi_M = \varphi \otimes \varphi_{\mathfrak{M}}$ and

$$\mathrm{Fil}^r M = \mathrm{Fil}^r \varphi^* \mathfrak{M} = \{x \in \varphi^* \mathfrak{M} | (1 \otimes \varphi)(x) \in E^r \mathfrak{M}\}.$$

It is clear from these descriptions that the restriction of φ_M to $\mathrm{Fil}^r M$ has image contained in $\varphi(E)^r M$, so we may and do define $\varphi_{M,r} := \varphi(E)^{-r} \varphi_M$ on $\mathrm{Fil}^r M$. We then set

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, A_{\inf}).$$

Lemma 4.1. *With notation as above*

  (1) *There is a natural isomorphism $\underline{T}_{\mathfrak{S}}(M) \simeq T_{\mathfrak{S}}(\mathfrak{M})$ of $\mathbf{Z}_p[G_\infty]$-modules;*
  (2) *There is an isomorphism of functors $\underline{T}_{\mathfrak{S}} \simeq \underline{T}_{\mathrm{cris}} \circ (\mathscr{M})$ on $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$.*

Proof. By [23][Lemma 3.3.4], there is a natural isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \underline{T}_{\mathrm{cris}}(\mathscr{M}(\mathfrak{M}))$ of $\mathbf{Z}_p[G_\infty]$-modules, so it suffices to prove (1). Using the relation $\varphi_{\star,r}(E^r x) = \varphi_\star(x)$ for $\star = M, A_{\inf}$, one shows that there is a caononcal map

(4.3) $\underline{T}_{\mathfrak{S}}(M) = \mathrm{Hom}_{\mathfrak{S},\mathrm{Fil}^r,\varphi_r}(M, A_{\inf}) \longrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(M, A_{\inf})$

of $\mathbf{Z}_p[G_\infty]$-modules that is visibly injective. We claim it is surjective as well, and hence an isomorphism. To see this, let $f \in \mathrm{Hom}_{\mathfrak{S},\varphi}(M, A_{\inf})$ and $x \in \mathrm{Fil}^r M$ be arbitrary. Again using the above relation, we compute

$$\varphi_{A_{\inf}}(f(x)) = f(\varphi_M(x)) = f(\varphi_{M,r}(E^r x)) = f(\varphi(E^r)\varphi_{M,r}(x)) = \varphi(E)^r f(\varphi_{M,r}(x)),$$

so recalling that $\varphi_{A_{\inf}}$ is an *automorphism* of $A_{\inf}$ we conclude that $f(x) = E^r \varphi_{A_{\inf}}^{-1} f(\varphi_{M,r}(x))$ and f carries $\mathrm{Fil}^r M$ into $\mathrm{Fil}^r A_{\inf}$. Written another way, this last equality reads

$$f(\varphi_{M,r}(x)) = \varphi_{A_{\inf}}(E^{-r} f(x)) = \varphi_{A_{\inf},r}(f(x))$$

and f is compatible with φ_r's. This shows that (4.3) is indeed an isomorphism as claimed.

To complete the proof, it now suffices to exhibit a natural isomorphism of $\mathbf{Z}_p[G_\infty]$-modules

(4.4) $T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, A_{\inf}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S},\varphi}(\varphi^* \mathfrak{M}, A_{\inf}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(M, A_{\inf})$.

To do this, for $f \in T_{\mathfrak{S}}(\mathfrak{M})$ we define $\iota(f) \in \mathrm{Hom}_{\mathfrak{S}}(\varphi^*\mathfrak{M}, A_{\mathrm{inf}})$ by

$$\iota(f)(\sum_i a_i \otimes m_i) := \sum_i a_i \varphi(f(m_i))$$

Since f is compatible with Frobenius, it is straightforward to see that the same is true of $\iota(f)$, so ι induces a map (4.4). Using the fact that $\varphi_{A_{\mathrm{inf}}}$ is bijective, one then checks easily that this map is an isomorphism as claimed. $\qquad\square$

In order to use the category of Breuil modules to study G_K representations (rather than just G_∞-representations), we require the additional structure of a monodromy operator. Let V be a crystalline representation with Hodge-Tate weights in $\{0, \ldots, r\}$ and $T \subset V$ a G_K-stable $\mathbf{Z}_p$-lattice. We denote $T^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p)$ the $\mathbf{Z}_p$-linear dual of T and put $V^\vee := T^\vee[1/p]$. For ease of notation, we write $D := D_{\mathrm{cris}}(V^\vee)$ for the associated filtered (φ, N)-module; of course $N_D = 0$ as V is crystalline.

By [5], we can functorially promote D to a *filtered (φ, N)-module* $\mathscr{D}(V) = (\mathscr{D}, \{\mathrm{Fil}^j\,\mathscr{D}\}_j, \varphi_{\mathscr{D}}, N_{\mathscr{D}})$ over $S[1/p]$ by defining

- $\mathscr{D} := S \otimes_{W(k)} D$ with $\varphi_{\mathscr{D}} := \varphi_S \otimes \varphi_D$
- $N_{\mathscr{D}} := N_S \otimes \mathrm{id} + \mathrm{id} \otimes N_D = N_S \otimes \mathrm{id}$, where $N_S : S \to S$ is the unique continuous W-linear derivation with $N(u) = -u$.
- $\mathrm{Fil}^j\,\mathscr{D}$ is defined inductively by setting $\mathrm{Fil}^0\,\mathscr{D} := \mathscr{D}$ and

$$\mathrm{Fil}^j\,\mathscr{D} = \{x \in \mathscr{D} | N(x) \in \mathrm{Fil}^{j-1}\,\mathscr{D},\ f_{\pi_0}(x) \in \mathrm{Fil}^j D_K\}$$

  where $f_{\pi_0} : \mathscr{D} \to D_K := \mathscr{O}_K \otimes_{W(k)} D$ is the projection induced by the map $f_{\pi_0} : S \twoheadrightarrow \mathscr{O}_K$ sending u to π_0.

The reader can consult the precise definition of filtered (φ, N)-modules over $S[1/p]$ in [23], which we do not need here. Following [6], we introduce:

Definition 4.2. *A strongly divisible S-lattice $\mathscr{M}$ inside $\mathscr{D} = \mathscr{D}(V)$ is a finite free S-submodule $\mathscr{M} \subset \mathscr{D}$ that is stable under $\varphi_{\mathscr{D}}$ and satisfies*

- $\mathscr{M}[1/p] = \mathscr{D}$;
- $\varphi_{\mathscr{D}}(\mathrm{Fil}^r\,\mathscr{M}) \subset p^r \mathscr{M}$ *where* $\mathrm{Fil}^r\,\mathscr{M} = \mathscr{M} \cap \mathrm{Fil}^r\,\mathscr{D}$;
- $N_{\mathscr{D}}(\mathscr{M}) \subset \mathscr{M}$.

Assuming $r \leq p - 2$, Breuil constructs a functor T_{st} on the category of strongly divisible lattices $\mathscr{M}$ in $\mathscr{D}$ with the property that $T_{\mathrm{st}}(\mathscr{M}) \subset V$ is a G_K-stable $\mathbf{Z}_p$-lattice. We refer the reader to [6] or [23] for details. The following theorem, synthesized from [6], [18], and [23], summarizes the relations between Breuil–Kisin modules, strongly divisible S-lattices, and lattices in Galois representations:

Theorem 4.3. *Let V be a crystalline G_K-representation with Hodge-Tate weights in $\{0, \ldots, r\}$ and $T \subset V$ a G_K-stable $\mathbf{Z}_p$-lattice. Then*

(1) *There exists a unique filtered Kisin module $M(T)$ of height r with $\underline{T}_{\mathfrak{S}}(M(T)) \simeq T^\vee|_{G_\infty}$.*

(2) *There exists an $S[1/p]$-linear isomorphism $\alpha_S : \underline{\mathscr{M}}(M(T))[1/p] \simeq \mathscr{D}(V)$ which is compatible with φ and filtrations.*

(3) *If $r \leq p - 2$, then the functor T_{st} induces an anti-equivalence between the category of strongly divisible S-lattices and the category of G_K-stable $\mathbf{Z}_p$-lattices T in crystalline G_K-representations with Hodge-Tate weights in $\{0, \ldots, r\}$.*

(4) *In the situation of (3), let $\mathscr{M}(T)$ be the strongly divisible S-lattice with $T_{\mathrm{st}}(\mathscr{M}) \simeq T^\vee$. Then there is a natural isomorphism $\underline{\mathscr{M}}(M(T)) \simeq \mathscr{M}(T)$ in $\mathrm{Mod}_S^{\varphi, r}$.*

Proof. Consider the version of the theorem with the classical Breuil–Kisin module $\mathfrak{M}(T)$ in place of its filtered counterpart $M(T)$ and the functor $T_{\mathfrak{S}}$ in place of $\underline{T}_{\mathfrak{S}}$. In this scenario, (1) is proved in [18], while (2) is proved in [23][§3.2]. We remark that the constructions of the filtrations on $\mathbf{D}(V)$ and $\underline{\mathscr{M}}(\mathfrak{M}(T))$ are very different, and that these two results have no restriction on r and hold more generally in the context of semistable G_K-representations. Statements (3) and (4) of this variant of Theorem 4.3 are the main results of [23], and also hold more generally for semistable V. Now by Lemma 4.1, we have $T_{\mathfrak{S}}(\mathfrak{M}(T)) \simeq \underline{T}_{\mathfrak{S}}(M(T))$ and $\underline{\mathscr{M}}(M(T)) = \underline{\mathscr{M}}(\mathfrak{M}(T))$, which, thanks to Remark 3.2, then gives our version of the theorem. $\qquad\square$

For future use, let us record a refinement of statements (1) and (2) of Theorem 4.3. Fix a G_K-stable $\mathbf{Z}_p$-lattice in a crystalline G_K-representation V and for notational ease put $\mathfrak{M} := \mathfrak{M}(T)$, and $\mathscr{D} := \mathscr{D}(V)$. As in [22][§3], one shows that $T_{\mathfrak{S}}$ induces a natural injection

$$A_{\inf} \otimes_{\mathfrak{S}} \mathfrak{M} \hookrightarrow T^{\vee} \otimes_{\mathbf{Z}_p} A_{\inf}$$

that intertwines $\varphi_{A_{\inf}} \otimes \varphi_{\mathfrak{M}}$ with $\mathrm{id} \otimes \varphi_{A_{\inf}}$ and $g \otimes \mathrm{id}$ with $g \otimes g$ for $g \in G_{\infty}$. Writing $M := \varphi^* \mathfrak{M}$ for the associated filtered Breuil–Kisin module, we deduce from Lemma 4.1 a similar injection

$$(4.5) \qquad\qquad \iota_{\mathfrak{S}} : A_{\inf} \otimes_{\mathfrak{S}} M \hookrightarrow T^{\vee} \otimes_{\mathbf{Z}_p} A_{\inf}$$

that is likewise comptible with the actions of φ and G_{∞}. The construction of the isomorphism α_S of (2) given in [23] then shows that the following diagram is commutative:

$$(4.6)$$

$$
\begin{array}{ccccc}
B_{\mathrm{cris}}^+ \otimes_{S[1/p]} \underline{\mathscr{M}}(M)[1/p] & = & B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} M & \xrightarrow{B_{\mathrm{cris}}^+ \otimes \iota_{\mathfrak{S}}} & T^{\vee} \otimes_{\mathbf{Z}_p} B_{\mathrm{cris}}^+ \\
{\scriptstyle B_{\mathrm{cris}}^+ \otimes \alpha_S}\Big\downarrow & & & & \Big\| \\
B_{\mathrm{cris}}^+ \otimes_{S[1/p]} \mathscr{D} & = & B_{\mathrm{cris}}^+ \otimes_{W(k)[1/p]} D & \xrightarrow[\iota_D]{} & V^{\vee} \otimes_{\mathbf{Z}_p} B_{\mathrm{cris}}^+
\end{array}
$$

where ι_D is the canonical injection arising from the very definition of $D = D_{\mathrm{cris}}(V^{\vee})$. Via this diagram, we henceforth regard $\mathfrak{M} \subset M \subset \mathscr{M} = \underline{\mathscr{M}}(M) \subset \mathscr{D}$ as submodules of $B_{\mathrm{cris}}^+ \otimes_{W(k)[1/p]} D$.

For use in the following section, we close this discussion with a brief review of *Breuil-Kisin-Fargues* modules, adapted from [4, §4.3]. Let F denote the fraction field of R.

Definition 4.4. A *Breuil–Kisin–Fargues module* is a pair $(\widetilde{M}, \varphi_{\widetilde{M}})$ where $\widetilde{M}$ is a finitely presented $A_{\inf}$-module with the property that $\widetilde{M}[1/p]$ is free over $A_{\inf}[1/p]$ and $\varphi_{\widetilde{M}} : \widetilde{M}[\frac{1}{E(u)}] \simeq \widetilde{M}[\frac{1}{\varphi(E(u))}]$ is a $\varphi_{A_{\inf}}$-semilinear isomorphism. Morphisms of Breuil–Kisin–Fargues modules are φ-compatible $A_{\inf}$-module homomorphisms.

Functorially associated to any Breuil–Kisin–Fargues module $(\widetilde{M}, \varphi_{\widetilde{M}})$ is a pair (L, Ξ) given by

$$L := (\widetilde{M} \otimes_{A_{\inf}} W(F))^{\varphi=1} \quad \text{and} \quad \Xi := \widetilde{M} \otimes_{A_{\inf}} B_{\mathrm{dR}}^+ \subset L \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}.$$

One proves (see [24] and the discussion in [4, §4.3]) that L is a finite free $\mathbf{Z}_p$-module and Ξ is a B_{dR}^+-lattice inside $L \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$, and that the functor $(\widetilde{M}, \varphi_{\widetilde{M}}) \rightsquigarrow (L, \Xi)$ is an equivalence between the category of finite free Breuil–Kisin–Fargues modules and the category of such pairs (L, Ξ).

Now let V be a crystalline G_K-representation with Hodge–Tate weights in $\{0, \ldots, r\}$, and let $T \subset V$ be a G_K-stable $\mathbf{Z}_p$-lattice. Let $M(T)$ be the filtered Breuil–Kisin module associated to T as in Theorem 4.3.

Corollary 4.5. $A_{\inf} \otimes_{\mathfrak{S}} M(T)$ *is the Breuil-Kisin-Fargues module corresponding to the pair* $(T^{\vee}, \Xi)$, *where* $\Xi := M(T) \otimes_{\mathfrak{S}} B_{\mathrm{dR}}^+ = D \otimes_{W(k)[1/p]} B_{\mathrm{cris}}^+ \subset T^{\vee} \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$.

Proof. This is essentially [4][Prop. 4.34], except that we use the contravariant functors $\underline{T}_{\mathfrak{S}}$ and $T_{\mathfrak{S}}$ from (filtered) Breuil–Kisin modules to Galois lattices. It is straightforward to translate between the version in *loc. cit.* and ours, as follows: It is clear that $\widetilde{M} := A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M(T)$ is a Breuil-Kisin-Fargues module. By [22] [Thm 3.2.2], the cokernel of the map $\iota_{\mathfrak{S}}$ in (4.5) is killed by $\varphi(\mathfrak{t})^r$, where $\mathfrak{t}$ is a certain element of $W(R)$ satisfying $\varphi(\mathfrak{t}) = E\mathfrak{t}$ (we note here that our map $\iota_{\mathfrak{S}}$ is the φ-twist of the map $\hat{\iota}$ in [22, 3.2.1]). Since $\mathfrak{t}$ is a unit of $W(F)$, we conclude that the scalar extension $W(F) \otimes_{A_{\mathrm{inf}}} \iota_{\mathfrak{S}} : W(F) \otimes_{\mathfrak{S}} M \to T^\vee \otimes_{\mathbf{Z}_p} W(F)$ is indeed an isomorphism. Passing to φ-invariants on both sides, we arrive at an isomorphism $T^\vee = (W(F) \otimes_{A_{\mathrm{inf}}} \widetilde{M})^{\varphi=1}$. $\qquad\square$

5. Crystalline cohomology

Let $\mathcal{X}$ be a smooth and proper formal scheme over $\mathscr{O}_K$ with (rigid analytic) generic fiber $X = \mathcal{X}_K$ over K, and put $\mathcal{X}_0 := \mathcal{X} \times_{\mathscr{O}_K} \mathscr{O}_K/(p)$ and $\mathcal{X}_k := \mathcal{X} \times_{\mathscr{O}_K} k$. For each nonnegative integer i, define

$$\mathscr{M}^i := H^i_{\mathrm{cris}}(\mathcal{X}_0/S) \text{ and } \mathscr{D}^i := \mathscr{M}^i[1/p]$$

which are naturally S and $S[1/p]$ modules, respectively, that are each equipped with a semilinear Frobenius endomorphism φ.

Let $V^i := H^i_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$. By [4, Theorem 1.1], V^i is a crystalline G_K-representation with Hodge-Tate weights in $\{-i, \ldots, 0\}$. Write $D^i := D_{\mathrm{cris}}((V^i)^\vee)$ for the filtered (φ, N)-module associated to the dual of V^i; of course, $N_D = 0$ as V^i is crystalline. By the C_{cris} comparison theorem [4, Theorem 1.1], we have $D^i \simeq H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$, compatibly with φ-actions and with filtrations after extending scalars to K. Let $\widetilde{\mathscr{D}}^i = \mathscr{D}((V^i)^\vee)$ be the filtered (φ, N)-module over S attached to $(V^i)^\vee$ as above Definition 4.2.

Consider the natural projection $q : S \twoheadrightarrow W(k)$ defined by $q(f(u)) = f(0)$ for $f(u) \in S$. This induces a natural map $\mathscr{M}^i \to H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ which we again denote by q.

Proposition 5.1. *There is a unique section* $s : H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] \to \mathscr{D}^i = H^i_{\mathrm{cris}}(\mathcal{X}_0/S)[1/p]$ *of* $q[1/p]$ *satisfying*

    (1) *s is φ-equivariant;*
    (2) *The map $S \otimes_{W(k)} H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] \to \mathscr{D}^i$ induced by s is an isomorphism.*

Identifying $H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] = D^i$ then gives a φ-equivariant isomorphism

$$(5.1) \qquad \qquad \mathscr{D}^i \simeq S \otimes_{W(k)} D^i =: \widetilde{\mathscr{D}}^i.$$

Remark 5.2. The Proposition is known when $\mathcal{X}$ is a smooth proper *scheme* by [17, Lemma 5.2] (*cf.* [26, Prop. 4.4.6]). We thank Yichao Tian for pointing out to us the possibility of extending Proposition 5.1 to the setting of formal schemes via the method of proof of [4, Prop. 13.9].

Proof. The following is a variant of the proof of [4, Prop. 13.9] obtained by replacing A_{cris} with S and making some necessary modifications to the argument. Let $S_{(n)}$ be the p-adic completion of the PD-envelope of $W(k)[u_n]$ with respect to $(E(u_n))$; note that $S_{(n)}$ and S_n are different if $n > 0$. There is an evident inclusion $S \hookrightarrow S_{(n)}$, and the Frobenius on S uniquely extends to $\varphi : S_{(n)} \to S_{(n)}$. Moreover, the self-map φ^n of $S_{(n)}$ induces a $W(k)$-semilinear isomorphism $\varphi^n : S_{(n)} \simeq S$. Consider the projection $S_{(n)} \twoheadrightarrow \mathscr{O}_{K_n}/(\pi_n^e)$ given by $u_n \to \pi_n$. This is a PD-thickening, and the isomorphism $\varphi^n : S_{(n)} \simeq S$ is compatible with the isomorphism $\varphi^n : \mathscr{O}_{K_n}/(\pi_n^e) \simeq \mathscr{O}_K/(\pi^e) = \mathscr{O}_K/(p)$ sending x to x^{p^n}, so $\varphi^n : S_{(n)} \simeq S$ is a morphism of divided power thickenings. Writing $\mathcal{X}_{(n)} := \mathcal{X} \times_{\mathscr{O}_K} \mathscr{O}_{K_n}/(\pi_n^e)$, we thus have the following isomorphisms by base change for crystalline cohomology:

$$(5.2) \quad H^i_{\mathrm{cris}}(\mathcal{X}_{(n)}/S_{(n)}) \otimes_{S_{(n)}, \varphi^n} S \simeq H^i_{\mathrm{cris}}(\mathcal{X}_{(n)} \times_{\mathscr{O}_{K_n}/(\pi_n^e), \varphi^n} \mathscr{O}_K/(p)/S) \simeq H^i_{\mathrm{cris}}(\mathcal{X}_0/S) \otimes_{S, \varphi^n} S.$$

On the other hand, if n is large enough (any n with $p^n \geq e$ will do), we have $\mathcal{X}_{(n)} \simeq \mathcal{X}_k \times_k \mathscr{O}_{K_n}/(\pi_n^e)$ because the canonical map $\mathscr{O}_K \to \mathscr{O}_{K_n}/(\pi_n^e)$ factors through $k = \mathscr{O}_K/(\pi)$. For such n, the inclusion $W(k) \hookrightarrow S_{(n)}$ is then a PD-morphism over $k \to \mathscr{O}_{K_n}/(\pi_n^e)$ so by base change and (5.2) we find

$$(5.3) \qquad H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k),\varphi^n} S \simeq H^i_{\mathrm{cris}}(\mathcal{X}_{(n)}/S_{(n)}) \otimes_{S_{(n)},\varphi^n} S \simeq H^i_{\mathrm{cris}}(\mathcal{X}_0/S) \otimes_{S,\varphi^n} S.$$

Composing (5.3) with the map $\varphi^n \otimes 1 : H^i_{\mathrm{cris}}(\mathcal{X}_0/S) \otimes_{S,\varphi^n} S \to H^i_{\mathrm{cris}}(\mathcal{X}_0/S)$, we obtain a map $s_n : H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k)) \to H^i_{\mathrm{cris}}(\mathcal{X}/S)$ that is φ-equivariant and has the property that $q \circ s_n$ is simply $\varphi^n : H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k)) \to H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))$. Since $\varphi^n : H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] \to H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ is bijective, we may finally define $s := s_n[1/p] \circ \varphi^{-n} : H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] \to \mathscr{D}^i$, which by construction is a φ-equivariant section of $q[1/p]$.

To show that the map $s(D^i) \otimes_{W(k)} S \to \mathscr{D}^i$ is bijective as claimed in (2), it suffices to show that $\varphi^n \otimes 1 : \mathscr{D}^i \otimes_{S,\varphi^n} S \to \mathscr{D}^i$ is bijective. Since the identification $D \otimes_{W(k),\varphi^n} S \simeq \mathscr{D}^i \otimes_{S,\varphi^n} S$ of (5.3) is compatible with φ, and the map $\varphi^n \otimes \varphi^n : D \otimes_{W(k),\varphi^n} S \to D \otimes_{W(k),\varphi^n} \varphi^n(S)$ is bijective, it follows that

$$(5.4) \qquad \varphi^n \otimes \varphi^n : \mathscr{D}^i \otimes_{S,\varphi^n} S \xrightarrow{\varphi^n \otimes 1} \mathscr{D}^i \otimes_S S \xrightarrow{1 \otimes \varphi^n} \mathscr{D}^i \otimes_{S,\varphi^n} \varphi^n(S)$$

is a bijection as well. It is obvious that the second map in (5.4) is bijective, because $\varphi : S \to \varphi(S)$ is an ring isomorphism. We conclude that $\varphi^n \otimes 1 : \mathscr{D}^i \otimes_{S,\varphi^n} S \to \mathscr{D}^i$ is bijective, as desired.

That s is unique is standard: if there exists another φ-equivariant section s', then for any $x \in D^i$ we have $(s - s')(x) \in \mathrm{Ker}(q)\mathscr{D}^i$. As $\varphi : D^i \to D^i$ is a bijection, for all $m > 0$ we may then write $x = \varphi^m(y_m)$ for suitable y_m. But then

$$(s - s')(x) = (s - s')(\varphi^m(y_m)) = \varphi^m((s - s)(y_m)) \in \varphi^m(\mathrm{Ker}(q))\mathscr{D}^i,$$

and this forces $(s - s')(x) = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using Proposition 5.1, we henceforth identify $\mathscr{D}^i$ with $\widetilde{\mathscr{D}}^i$ via (5.1). If we further assume that $\mathscr{M}^i$ is torsion free, we may view $\mathscr{M}^i$ as an S-submodule of $\widetilde{\mathscr{D}}^i$, and we then *define* $\mathrm{Fil}^i \mathscr{M}^i := \mathrm{Fil}^i \widetilde{\mathscr{D}}^i \cap \mathscr{M}^i$.

Put $T^i := H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)/\mathrm{tor}$; it is a G_K-stable $\mathbf{Z}_p$-lattice inside the crystalline representation V^i so has an associated filtered Breuil–Kisin module $M^i := M^i((T^i)^\vee)$ via Theorem 4.3 (1). Then Theorem 4.3 (2) provides an isomorphism of $S[1/p]$-modules

$$\alpha_S : \underline{\mathscr{M}}(M^i)[1/p] \simeq \widetilde{\mathscr{D}}^i$$

that is compatible φ and Fil^i (we forget the N-structure here). Since we have identified $\widetilde{\mathscr{D}}^i$ with $\mathscr{D}^i$ in the above, we arrive an isomorphism of $S[1/p]$-modules

$$(5.5) \qquad \iota : \underline{\mathscr{M}}(M^i)[1/p] \simeq \mathscr{D}^i = \mathscr{M}^i[1/p] \simeq S \otimes_{W(k)} D^i$$

compatible with φ and Fil^i. We reiterate that, as explained below (4.6), we will regard all modules involved in this discussion as submodules of $B^+_{\mathrm{cris}} \otimes_{W(k)[1/p]} D^i$ via ι.

Our aim is to prove:

Theorem 5.3. *Let i be a nonnegative integer with $i < p - 1$, and assume that $H^i_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ and $H^{i+1}_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ are torsion free. Then the following hold:*

 (1) *$T^i := H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ is also torsion-free.*
 (2) *$\mathscr{M}^i := H^i_{\mathrm{cris}}(\mathcal{X}_0/S)$ is a strongly divisible S-lattice in $\mathscr{D}^i$ and $T_{\mathrm{st}}(\mathscr{M}^i) \simeq T^i$.*
 (3) *There is a natural isomorphism of Breuil–Kisin modules $\underline{M}(\mathscr{M}^i) \simeq M^i((T^i)^\vee)$.*

To prove Theorem 5.3, we must first recall [4, Thm 1.8], which provides a perfect complex $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ of A_{inf}-modules with a φ-linear map $\varphi : R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \to R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ such that

(1) $\widetilde{M}^i := H^i(R\Gamma_{A_{\inf}}(\mathcal{X}))$ is a Breuil–Kisin–Fargues module.

(2) $H^i(R\Gamma_{A_{\inf}}(\mathcal{X}) \otimes_{A_{\inf}}^{\mathbb{L}} A_{\mathrm{cris}}) \simeq H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}}) \simeq A_{\mathrm{cris}} \otimes_S \mathscr{M}^i$ as φ-modules over A_{cris}.

(3) $H^i(R\Gamma_{A_{\inf}}(\mathcal{X}) \otimes_{A_{\inf}}^{\mathbb{L}} W(\overline{k})) \simeq H^i_{\mathrm{cris}}(\mathcal{X}_{\overline{k}}/W(\overline{k}))$ as φ-modules over $W(\overline{k})$.

(4) $H^i(R\Gamma_{A_{\inf}}(\mathcal{X}) \otimes_{A_{\inf}} W(F)) \simeq H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} W(F)$

We advise the reader that (4) is slightly different from the comparison isomorphism found in [4], where the period ring $A_{\inf}[1/\mu]$ is used in place of $W(F)$. Here $\mu = [\varepsilon] - 1$ for $\varepsilon = (\zeta_{p^n})_{n \geq 0} \in R$ with $\{\zeta_{p^n}\}$ a compatible system of primitive p^n-th root of unity. However, it is not difficut to see that $W(F)$ is flat over $A_{\inf}[1/\mu]$ (*cf.* the proof of Lemma 5.6) and then (4) follows easily from the comparison isomorphism found in [4, Theorem 1.8].

From these facts we deduce a natural map of φ-modules over A_{cris}:

$$(5.6) \qquad\qquad \widetilde{\iota} : A_{\mathrm{cris}} \otimes_{A_{\inf}} \widetilde{M}^i \to H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})$$

Lemma 5.4. *There is a natural, φ-compatible isomorphism of $A_{\inf}$-modules*

$$\alpha : A_{\inf} \otimes_{\mathfrak{S}} M^i[1/p] \simeq \widetilde{M}^i[1/p]$$

with the property that the following diagram commutes:

$$(5.7)$$

$$
\begin{array}{ccc}
B^+_{\mathrm{cris}} \otimes_{A_{\inf}} \widetilde{M}^i & \xrightarrow{\widetilde{\iota}[1/p]} & H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})[1/p] \\
{\scriptstyle B^+_{\mathrm{cris}} \otimes \alpha} \big\uparrow & & \big\uparrow {\scriptstyle \wr} \\
B^+_{\mathrm{cris}} \otimes_{\mathfrak{S}} M^i & \xrightarrow[B^+_{\mathrm{cris}} \otimes \iota]{\sim} & B^+_{\mathrm{cris}} \otimes_S \mathscr{M}^i
\end{array}
$$

In particular, $\widetilde{\iota}[1/p]$ is an isomorphism.

Proof. This follows the main result of [4] cited above. Indeed, by Definition 4.4 and (1) above, each $\widetilde{M}^i[1/p]$ is a finite and *free* $A_{\inf}[1/p]$-module, so the derived comparison isomorphisms (2) and (3) yield comparison isomorphisms on the individual cohomology groups with p-inverted. In particular, the natural induced maps

$$\widetilde{\iota}[1/p] : A_{\mathrm{cris}} \otimes_{A_{\inf}} \widetilde{M}^i[1/p] \longrightarrow H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})[1/p]$$

and

$$\widetilde{M}^i[1/p] \otimes_{A_{\inf}} W(\overline{k}) \longrightarrow H^i_{\mathrm{cris}}(\mathcal{X}_{\overline{k}}/W(\overline{k}))[1/p] = W(\overline{k}) \otimes_{W(k)} D^i$$

are φ-compatible isomorphisms. Furthermore, as explained following [4, Thm. 1.8], these mappings are compatible with the canonical projection $q : \widetilde{M}^i[1/p] \to \widetilde{M}^i[1/p] \otimes_{A_{\inf}} W(\overline{k})$ and the projection $q' : H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})[1/p] \to H^i_{\mathrm{cris}}(\mathcal{X}_{\overline{k}}/W(\overline{k}))[1/p]$ arising via the compatibility of crystalline cohomology with PD-base change in the sense that the obvious diagram commutes. It follows that $\widetilde{\iota}$ induces a φ-compatible isomorphism

$$A_{\mathrm{cris}} \otimes_{A_{\inf}} \widetilde{M}^i[1/p] \xrightarrow[\widetilde{\iota}[1/p]]{\sim} H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})[1/p] \xleftarrow[\beta]{\sim} B^+_{\mathrm{cris}} \otimes_{W(k)[1/p]} \overline{D}^i$$

where $\overline{D}^i := H^i(\mathcal{X}_{\overline{k}}/W(\overline{k}))[1/p]$ and the isomorphism β is induced by constructing a φ-equivariant section $s : \overline{D}^i \hookrightarrow H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})[1/p]$ to the projection map q'; we note that such a section s exists and is necessarily unique, whence β is unique as well (see [4, Prop. 13.9] and Proposition 5.1). Using the isomorphisms $\widetilde{\iota}[1/p]$ and β, we then regard $\widetilde{M}^i[1/p]$ and $H^i_{\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}})$ as submodules of $B^+_{\mathrm{cris}} \otimes_{W(k)[1/p]} D^i$. As explained in the beginning of this section, we use tha map ι

of (5.5) to regard both M^i and $\mathscr{D}^i = H^i_{\mathrm{cris}}(\mathcal{X}_0/S)[1/p]$ as submodules of $B^+_{\mathrm{cris}} \otimes_{W(k)[1/p]} D^i$. Thus, working entirely inside $B^+_{\mathrm{cris}} \otimes_{W(k)[1/p]} D^i$, it now suffices to prove that $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M^i[1/p] = \widetilde{M}^i[1/p]$. By [4][Prop. 4.13], there exists an exact sequence of A_{inf}-modules

$$(5.8) \qquad 0 \longrightarrow \widetilde{M}^i_{\mathrm{tor}} \longrightarrow \widetilde{M}^i \longrightarrow \widetilde{M}^i_{\mathrm{fr}} \longrightarrow \overline{M}^i \longrightarrow 0.$$

where $\widetilde{M}^i_{\mathrm{tor}}$ is killed by power of p, the term $\widetilde{M}^i_{\mathrm{fr}}$ free of finite rank over A_{inf}, and $\overline{M}^i$ is killed by some power of the ideal (u, p). We claim that $\widetilde{M}^i_{\mathrm{fr}}$ is the Breuil–Kisin–Fargues module corresponding to the pair (T^i, Ξ), with $\Xi := B^+_{\mathrm{dR}} \otimes_{A_{\mathrm{inf}}} \widetilde{M}^i_{\mathrm{fr}} \subset T^i \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$. To see this, we apply $\otimes_{A_{\mathrm{inf}}} W(F)$ to the exact sequence (5.8) and, using Lemma 5.6 (1) below with the fact that $\overline{M}^i$ is killed by a power of $u \in W(F)^\times$, we deduce an exact sequence of $W(F)$-modules

$$(5.9) \qquad 0 \longrightarrow \widetilde{M}^i_{\mathrm{tor}} \otimes_{A_{\mathrm{inf}}} W(F) \longrightarrow \widetilde{M}^i \otimes_{A_{\mathrm{inf}}} W(F) \longrightarrow \widetilde{M}^i_{\mathrm{fr}} \otimes_{A_{\mathrm{inf}}} W(F) \longrightarrow 0 .$$

Writing $T_\star := (\widetilde{M}^i_\star \otimes_{A_{\mathrm{inf}}} W(F))^{\varphi=1}$ for $\star \in \{\mathrm{tor}, \emptyset, \mathrm{fr}\}$, Lemma 4.26 of [4] gives that $T_\star$ is a finite-type $\mathbf{Z}_p$-module and canonically identifies (5.9) with the exact sequence

$$(5.10) \qquad 0 \longrightarrow T_{\mathrm{tor}} \otimes_{A_{\mathrm{inf}}} W(F) \longrightarrow T \otimes_{A_{\mathrm{inf}}} W(F) \longrightarrow T_{\mathrm{fr}} \otimes_{A_{\mathrm{inf}}} W(F) \longrightarrow 0 .$$

From the very definition of $T_\star$, we have a sequence of $\mathbf{Z}_p$-modules

$$(5.11) \qquad 0 \longrightarrow T_{\mathrm{tor}} \longrightarrow T \longrightarrow T_{\mathrm{fr}} \longrightarrow 0 ,$$

whose scalar extension to $W(F)$ is the exact sequence (5.10). Thus, since $\mathbf{Z}_p \to W(F)$ is faithfully flat [25, Tag 0539], we conclude that (5.11) is *exact*. Now since $\widetilde{M}^i_{\mathrm{fr}}$ if finite free over A_{inf}, it is clear that T_{fr} is free over $\mathbf{Z}_p$, and we have $T \simeq H^i_{\mathrm{ét}}(X_{\overline{X}}, \mathbf{Z}_p)$ thanks to the comparison isomorphism $H^i(R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes_{A_{\mathrm{inf}}} W(F)) \simeq H^i_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} W(F)$ of [4, Theorem 1.8] recorded above and the fact that $A_{\mathrm{inf}} \to W(F)$ is flat, recorded in Lemma 5.6 (1) below. It follows at last that we have $(\widetilde{M}^i_{\mathrm{fr}} \otimes_{A_{\mathrm{inf}}} W(F))^{\varphi=1} =: T = H^i_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Z}_p)/\mathrm{tors} =: T^i$, which gives our claim. Now since we clearly have $\widetilde{M}^i_{\mathrm{fr}}[1/p] = \widetilde{M}^i[1/p]$, we may rewrite $\Xi = B^+_{\mathrm{dR}} \otimes_{A_{\mathrm{inf}}} \widetilde{M}^i_{\mathrm{fr}} = B^+_{\mathrm{dR}} \otimes_{W(k)[1/p]} D^i$. Since $M^i = M((T^i)^\vee)$, Corollary 4.5 then shows that $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M^i$ is the Breuil–Kisin–Fargues module corresponding to (T^i, Ξ). This yields the desired identification $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M^i = \widetilde{M}^i_{\mathrm{fr}}$ inside $B^+_{\mathrm{cris}} \otimes_{W(k)[1/p]} D^i$. $\qquad \square$

To proceed further, we will need:

Proposition 5.5. *Suppose that $\widetilde{M}^{i+1}$ is u-torsion free. Then (5.6) is an isomorphism.*

To prove this proposition, we first require some preparations. Put $F := \mathrm{Frac}(R)$ and note that $W(F)$ is a complete DVR with uniformizer p.

Lemma 5.6. *The following statements hold:*

    (1) $W(F)$ *is flat over A_{inf}.*

    (2) $W(F) \cap A_{\mathrm{inf}}[1/p] = A_{\mathrm{inf}}$.

    (3) *Let M be a finitely generated A_{inf}-module. The map $M \to W(F) \otimes_{A_{\mathrm{inf}}} M$ is injective if and only if M has no u-torsion.*

    (4) $A_{\mathrm{cris}}/(p^n)$ *is faithfully flat over $S/(p^n)$ for all $n \geq 1$.*

Proof. Let $A' := (A_{\mathrm{inf}})_{(p)}$ be the localization of A_{inf} at the prime ideal (p). Then A' is a local ring with uniformizer p and residue field $F = \mathrm{Frac}\, R$, so is a discrete valuation ring and in particular is noetherian. As a localization of A_{inf}, it is moreover flat over A_{inf}. One checks via the theory of strict p-rings that $W(F)$ is the p-adic completion of A', and hence flat over A' as the completion of *any* noetherian local ring is flat [25, Tag 00MB]. It follows that $W(F)$ is flat over A_{inf} giving (1).

To prove (2), suppose $x \in A_{\mathrm{inf}}[1/p] \cap W(F)$. Then

$$p^m x \in A_{\mathrm{inf}} \cap p^m W(F) = p^m W(R) = p^m A_{\mathrm{inf}}$$

by basic properties of Witt vectors, which gives $x \in A_{\mathrm{inf}}$.

Now M has no u-torsion if and only if the map $M \to A' \otimes_{A_{\mathrm{inf}}} M$ is injective. Since $M' := A' \otimes_{A_{\mathrm{inf}}} M$ is finitely generated over the noetherian ring A', we know that $\widehat{A'} \otimes_{A_{\mathrm{inf}}} M = W(F) \otimes_{A_{\mathrm{inf}}} M$ is the p-adic completion of M' thanks to [25, Tag 00MA]. Again, M' is a finitely generated module over the noetherian ring A', so the Krull intersection theorem [25, Tag 00IP] implies that the map from M' to its p-adic completion is injective, which gives (3).

To prove (4), let $S^{\mathrm{PD}} := \mathfrak{S}[\{E(u)^n/n!\}_{n\geq 1}]$ be the divided power envelop of $\mathfrak{S}$ with respect to the kernel of the surjection $\mathfrak{S} \twoheadrightarrow \mathscr{O}_K$ sending u to π_0, so that S is the p-adic completion of S^{PD}. We similarly write $A_{\mathrm{inf}}^{\mathrm{PD}}$ for the divided power envelope of A_{inf} with respect to $\ker(\theta_{\mathrm{inf}})$, so again A_{cris} is the p-adic completion of $A_{\mathrm{inf}}^{\mathrm{PD}}$. We claim that the natural map $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} S^{\mathrm{PD}} \to A_{\mathrm{inf}}^{\mathrm{PD}}$ is an isomorphism. Indeed, this follows from [25, Tag 07HD] once we check that $\mathfrak{S}/(p) \to A_{\mathrm{inf}}/(p)$ is flat and $\mathrm{Tor}_1^{\mathfrak{S}}(A_{\mathrm{inf}}, \mathfrak{S}/(p)) = 0$. As $\mathfrak{S}/(p) = k[\![u]\!]$ is a DVR and $A_{\mathrm{inf}}/(p) = R$ is torsion-free, the first is clear [25, Tag 0539], as is the second since $A_{\mathrm{inf}} = W(R)$ is p-torsion free. Thanks to [4][Lem. 4.30], the map $\mathfrak{S} \to A_{\mathrm{inf}}$ is flat, whence its scalar extension $S^{\mathrm{PD}} = \mathfrak{S} \otimes_{\mathfrak{S}} S^{\mathrm{PD}} \to A_{\mathrm{inf}} \otimes_{\mathfrak{S}} S_{\mathrm{PD}} = A_{\mathrm{inf}}^{\mathrm{PD}}$ is flat as well. This implies that the map $S/(p^n) = S^{\mathrm{PD}}/(p^n) \to A_{\mathrm{inf}}^{\mathrm{PD}}/(p^n) = A_{\mathrm{cris}}/(p^n)$ is flat for every $n \geq 1$. Since $S \to A_{\mathrm{cris}}$ is a local map of local rings we conclude that the flat maps $S/(p^n) \to A_{\mathrm{cris}}/(p^n)$ are faithfully flat. $\qquad\square$

Remark 5.7. Perhaps surprisingly, we do not know whether or not $S \to A_{\mathrm{cris}}$ is (faithfully) flat.

In what follows, for an A_{inf}-module M we will simply write $M_{W(F)}$ for the scalar extension $W(F) \otimes_{A_{\mathrm{inf}}} M$. Likewise, for a map $f : M \to M'$ of A_{inf}-modules, we write $f_{W(F)} := f \otimes 1 : M_{W(F)} \to M'_{W(F)}$ for the induced map of $W(F)$-modules.

Lemma 5.8. *Let $f : M \to M'$ be a map of A_{inf}-modules. Assume that*

(1) *M is finite and free over A_{inf},*
(2) *M' is u-torsion free,*
(3) *$N := \mathrm{Ker}(f)$ is a finitely generated A_{inf}-module.*

Then N is finite free over A_{inf}.

Proof. Replacing M' by $f(M)$, we may assume that f is surjective and hence also that M' is finitely generated. Lemma 5.6 then gives a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_{W(F)} & \longrightarrow & M_{W(F)} & \longrightarrow & M'_{W(F)} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N[1/p] & \longrightarrow & M[1/p] & \longrightarrow & M'[1/p] & \longrightarrow & 0
\end{array}
$$

in which all vertical arrows, with the possible exception of the lower right arrow, are injective. Now we claim that $N = N[1/p] \cap N_{W(F)}$ inside $M_{W(F)}[1/p]$. To prove the claim, let $x \in N[1/p] \cap N_{W(F)} \subset M[1/p] \cap M_{W(F)}$ be arbitrary. Then from the diagram we see that $f[1/p](x) = f_{W(F)}(x) = 0$. On the other hand, using Lemma 5.6 (2) and our hypothesis that M is finite free over A_{inf}, we deduce that $x \in M[1/p] \cap M_{W(F)} = M$. Furthermore, since the upper vertical arrows are injective and $f_{W(F)}(x) = 0$, we must have $f(x) = 0$ and $x \in \ker(f) = N$ as claimed. From this claim it follows at once that the natural map $N/pN \to N_{W(F)}/pN_{W(F)}$ is *injective*. Since the target is a finite dimensional F-vector space and the source is a finitely generated R-module, we conclude that N/pN is a finite and *torsion-free* R-module, and hence a finitely generated submodule of a free R-module.[2] Now R is a (non-noetherian) valuation ring, and hence a Bézout domain, from which it follows that N/pN is a finite and *free* R-module [25, Tag 0ASU]. Let $x_1, \ldots, x_m$ be a R-basis of N/pN and choose lifts $\hat{x}_i \in N$. By Nakayama's Lemma, these lifts generate N as an A_{inf}-module. Now any nontrivial A_{inf}-relation $\sum \alpha_i \hat{x}_i = 0$ on these generators may be re-written as $p^j \sum \alpha_i' \hat{x}_i = 0$ for some nonnegative integer j, with at least one α_i' nonzero modulo p. But N is a submodule of the free module M, and hence torsion-free, from which it follows that $\sum \alpha_i' \hat{x}_i = 0$. But this relation reduces modulo p to a *non-trivial* relation on the x_i, contradicting the fact that N/pN is R-free. We conclude that the $\hat{x}_i$ freely generate N, as desired.   □

Proof of Proposition 5.5. Let $C_\bullet$ be a bounded complex of finite projective A_{inf}-modules that is quasi-isomorphic to the perfect complex $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$. As A_{inf} is a local ring, each term C_i is a finite, free A_{inf}-module. We then have $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes_{A_{\mathrm{inf}}}^{\mathbb{L}} A_{\mathrm{cris}} = C_\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$, and our goal is to prove that the canonical map

$$H^i(C_\bullet) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \to H^i(C_\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}})$$

is an isomorphism. Writing $d_j : C_j \to C_{j+1}$ for the given maps, we first show that the natural map

$$(5.12) \qquad\qquad A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \mathrm{Ker}(d_i) \to \mathrm{Ker}(A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} d_i)$$

is an isomorphism. To see this, consider the exact sequences of A_{inf}-modules

$$(5.13\mathrm{a}) \qquad\qquad 0 \longrightarrow \ker(d_i) \longrightarrow C_i \longrightarrow \mathrm{im}(d_i) \longrightarrow 0$$

$$(5.13\mathrm{b}) \qquad\qquad 0 \longrightarrow \mathrm{im}(d_i) \longrightarrow \ker(d_{i+1}) \longrightarrow \widetilde{\mathfrak{M}}^{i+1} \longrightarrow 0$$

Now C_i is finite over A_{inf}, so same is true of its quotient $\mathrm{im}(d_i)$, whence $\ker(d_{i+1})$ is finitely generated because the Breuil–Kisin–Fargues module $\widetilde{M}_{i+1}$ is. Likewise, $\ker(d_i)$ is finitely generated, and since C_i and C_{i+1} are finite free, it follows from Lemma 5.8 that $\ker(d_i)$ and $\ker(d_{i+1})$ are finite free— hence flat—A_{inf}-modules. Since A_{inf} and A_{cris} are both contained in field B_{dR}, an easy argument then shows that the inclusions $\ker(d_i) \hookrightarrow C_i$ and $\ker(d_{i+1}) \hookrightarrow C_{i+1}$ remain injective after tensoring with A_{cris}; in particular, the sequence (5.13a) remains exact after tensoring with A_{cris}.

In general, the sequence (5.13b) may not remain exact after tensoring with A_{cris}. However, using our hypothesis that $\widetilde{M}^{i+1}$ is u-torsion free and the fact that $\mathrm{im}(d_i)$ is finitely generated, Lemma 5.8 shows that $\mathrm{im}(d_i)$ is in fact finite and free as an A_{inf}-module. Arguing as above, we conclude that the sequence (5.13b) likewise remains exact after tensoring with A_{cris}. Since $\ker(d_{i+1}) \hookrightarrow C_{i+1}$ remains injective after tensoring with A_{cris}, it follows that the map $\mathrm{im}(d_i) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \to \mathrm{im}(d_i \otimes A_{\mathrm{cris}})$ is injective. An easy diagram chase now shows that the map (5.12) is an isomorphism as claimed.

---

[2]Indeed, an elementary argument shows that any finitely generated and torsion-free module over a commutative domain may be embedded as a submodule of a free module.

Now consider the following diagram

$$
\begin{array}{ccccccc}
A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \mathrm{im}(d_{i-1}) & \longrightarrow & A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \mathrm{Ker}(d_i) & \longrightarrow & A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \widetilde{\mathfrak{M}}^i & \longrightarrow & 0 \\
\downarrow & & \downarrow \wr & & \downarrow \widetilde{\iota} & & \\
0 \longrightarrow \mathrm{im}(A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} d_{i-1}) & \longrightarrow & \mathrm{Ker}(A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} d_i) & \longrightarrow & H^i_{\mathrm{cris}}(\mathcal{X}/A_{\mathrm{cris}}) & \longrightarrow & 0
\end{array}
$$

Since the first column is surjective by right-exactness of tensor product and we have just seen that the second column is an isomorphism, the Snake Lemma completes the proof that the map $\widetilde{\iota}$ of the third column is an isomorphism. $\qquad\square$

Remark 5.9. Our application of Proposition 5.5 will be to the proof of Theorem 5.3. Under the hypotheses of this theorem, one knows that $\widetilde{M}^j$ is in fact *free* of finite rank over A_{inf} for $j = i, i+1$ (see below), so for our purposes it would be enough to have the conclusion of Proposition 5.5 under the stronger hypothesis that $\widetilde{M}^{i+1}$ is free over A_{inf}. The following short proof of this variant was suggested to us by Bhargav Bhatt.

In the notation of the proof of Proposition 5.5, and putting $M := \widetilde{M}^j = H^j(C_\bullet)$, we first claim that the complex $M \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ is concentrated in homological degrees 0 and 1; that is, that $\mathrm{Tor}_i^{A_{\mathrm{inf}}}(M, A_{\mathrm{cris}}) = 0$ for $i \geq 2$. To see this, first note that M has bounded p-power torsion and, as a complex of A_{inf}-modules, is perfect thanks to Theorem 1.8 and Lemma 4.9 of [4]. It follows that the pro-systems $\{M/p^n M\}_n$ and $\{M \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(p^n)\}$ are pro-isomorphic, and that $M \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ is p-adically complete. We deduce isomorphisms

$$
M \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \simeq \varprojlim_n \left(M \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}/(p^n) \right) \simeq \varprojlim_n \left(M/p^n M \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}/(p^n)} A_{\mathrm{cris}}/(p^n) \right).
$$

The claim follows from the fact that $A_{\mathrm{cris}}/(p^n)$ has Tor-dimension 1 over $A_{\mathrm{inf}}/(p^n)$.[3]

Next, we claim that the cohomology groups $H^j(\tau^{>i+1} C_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}})$ vanish for $j \leq i$. More generally, suppose that $N_\bullet$ is any bounded complex with $\tau^{<0} N_\bullet = 0$ and $H^j(N_\bullet) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ concentrated in homological degrees 0 and 1. Then an easy induction argument on the number of nonzero terms in $N_\bullet$ shows that $\tau^{<1}(N_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}) = 0$, and hence that $H^0(N_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}})$—which is a quotient of this complex—is zero as well. Applying this with $N_\bullet = (\tau^{>i+1} C_\bullet)[-j]$ then gives the claimed vanishing.

Applying $\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ to the exact triangle

$$
\tau^{\leq i+1} C_\bullet \longrightarrow C_\bullet \longrightarrow \tau^{>i+1} C_\bullet
$$

and passing to the long exact sequence of cohomology modules thus yields an isomorphism

$$
H^i(\tau^{\leq i+1} C_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}) \simeq H^i(C_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}).
$$

On the other hand, applying $\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ to the exact triangle

$$
\tau^{\leq i} C_\bullet \longrightarrow \tau^{\leq i+1} C_\bullet \longrightarrow H^{i+1}(C_\bullet)[-i-1]
$$

---

[3]Indeed, one has the evident presentation $A_{\mathrm{inf}}/(p^n)\langle T \rangle \xrightarrow{T-\xi_n} A_{\mathrm{inf}}/(p^n)\langle T \rangle \twoheadrightarrow A_{\mathrm{cris}}/(p^n)$, where ξ_n is a generator of the principal ideal $\ker(A_{\mathrm{inf}}/(p^n) \twoheadrightarrow \mathscr{O}_{\overline{K}}/(p^n))$. From the very construction of the divided power polynomial algebra $A_{\mathrm{inf}}/(p^n)\langle T \rangle$, this is then a 2-term resolution by free—hence flat—modules, so [25, Tag 066F] gives the asserted Tor-dimension.

and passing to cohomology gives the short exact sequence

$$0 \longrightarrow H^i(\tau^{\leq i}C_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}) \longrightarrow H^i(\tau^{\leq i+1}C_\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}) \longrightarrow \mathrm{Tor}_1^{A_{\mathrm{inf}}}(H^{i+1}(C_\bullet), A_{\mathrm{cris}}) \longrightarrow 0$$

in which the first term is readily seen to be isomorphic to $H^i(C^\bullet) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ since we are taking "top degree" cohomology. Thus, when $\widetilde{M}^{i+1} = H^{i+1}(C_\bullet)$ is free over A_{inf} so the Tor vanishes, we deduce that (5.6) is an isomorphism, as desired.

Proof of Theorem 5.3. If j is *any* nonnegative integer with the property that $H^j_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ is torsion free, then Theorem 14.5 and Proposition 4.34 of [4] show

(1) $T = H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ is finite free of $\mathbf{Z}_p$-rank $d := \dim_{\mathbf{Q}_p} H^j_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$.
(2) $\widetilde{M}^j$ is finite free of rank d over A_{inf} and $\widetilde{M}^j \simeq A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M^j$ via the map α of Lemma 5.4.

Thus, our hypothesis that $H^j_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ is torsion-free for $j = i, i+1$ implies in particular that $\widetilde{M}^{i+1}$ is u-torsion free, and hence that the natural map

$$\widetilde{\iota} : A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \widetilde{M}^i \longrightarrow H^i(\mathcal{X}_{\mathscr{O}_{\overline{K}}/(p)}/A_{\mathrm{cris}}) = A_{\mathrm{cris}} \otimes_S \mathscr{M}^i$$

of (5.6) is an isomorphism thanks to Lemma 5.5. Thus, $A_{\mathrm{cris}} \otimes_S \mathscr{M}^i$ is a finite and free A_{cris}-module of rank d, so also $A_{\mathrm{cris}} \otimes_S \mathscr{M}^i/(p^n)$ is finite and free of rank d as an $A_{\mathrm{cris}}/(p^n)$-module. Using Lemma 5.6 (4) together with the facts that the property "finite projective" for modules descends along faithfully flat morphisms [25, Tag 058S] and finite projective implies free [25, Tag 00NZ] (see also [25, Tag 0593]), we deduce that $\mathscr{M}^i/(p^n)$ is finite and free over $S/(p^n)$, necessarily of rank d. Let $e_1, \ldots, e_d$ be a basis of $\mathscr{M}^i/(p^n)$ and choose lifts $\hat{e}_j \in \mathscr{M}^i$ of e_j. By Nakayama's Lemma, $\mathscr{M}$ is generated by $\hat{e}_j$. There can be no linear relations as $\mathscr{M}^i[1/p] = \mathscr{D}^i \simeq \widetilde{\mathscr{D}}^i = S \otimes_{W(k)} D^i$ is finite and free over $S[1/p]$ of rank $d = \dim_{W(k)[1/p]} D^i$. This proves that $\mathscr{M}^i$ is finite S-free.

Consider the isomorphism $\iota : \underline{\mathscr{M}}(M^i)[1/p] \simeq \mathscr{M}^i[1/p]$ of (5.5). Identifying $\underline{\mathscr{M}}(M^i)$ with its image in $\mathscr{D}^i = \mathscr{M}^i[1/p]$ under ι, our goal is then to prove that $\underline{\mathscr{M}}(M^i) = \mathscr{M}^i$ inside $\mathscr{D}^i$, and to do so it suffices to prove that $q(M^i) = q(\mathscr{M}^i)$ where $q : \mathscr{D} \to D$ is the canonical projection induced by reduction modulo $I_+S := S[1/p] \cap uK_0[\![u]\!]$. Extending q to a map

$$\widetilde{q} : B^+_{\mathrm{cris}} \otimes_S \mathscr{M} = B^+_{\mathrm{cris}} \otimes_{S[1/p]} \mathscr{D}^i \to W(\overline{k})[1/p] \otimes_{K_0} D$$

in the obvious way, due to Lemma 5.4 and our identifications, it is then enough to prove that

(5.14) $\widetilde{q}(A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \widetilde{M}^i) = \widetilde{q}(A_{\mathrm{cris}} \otimes_S \mathscr{M}^i),$

where $A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \widetilde{M}^i$ is viewed as a submodule of $B^+_{\mathrm{cris}} \otimes_S \mathscr{M}^i$ via (5.7). But $\widetilde{\iota}$ carries $A_{\mathrm{cris}} \otimes_{A_{\mathrm{inf}}} \widetilde{M}^i$ isomorphically on to $A_{\mathrm{cris}} \otimes_S \mathscr{M}^i$ as we have seen, so the desired equality (5.14) indeed holds.

Recalling that we have defined $\mathrm{Fil}^i \mathscr{M}^i := \mathrm{Fil}^i \mathscr{D}^i \cap \mathscr{M}^i$, that the map ι of (5.5) is compatible with filtrations, we find

(5.15) $\mathrm{Fil}^i \mathscr{M}^i = \mathrm{Fil}^i \mathscr{D}^i \cap \mathscr{M}^i = \mathrm{Fil}^i(\underline{\mathscr{M}}(M^i)[1/p]) \cap \mathscr{M}^i = \mathrm{Fil}^i(\underline{\mathscr{M}}(M^i)[1/p]) \cap \underline{\mathscr{M}}(M^i)$

via our identifications. From the construction of $\mathrm{Fil}^i \underline{\mathscr{M}}(M^i)$, it is easy to show that $\mathrm{Fil}^i \underline{\mathscr{M}}(M^i)$ is saturated as submodule of $\underline{\mathscr{M}}(M^i)$ Hence the right side of (5.15) coincides with $\mathrm{Fil}^i \underline{\mathscr{M}}(\mathfrak{M}^i)$. We conclude that the φ-compatible isomorphism of S-modules $\underline{\mathscr{M}}(\mathfrak{M}^i) \simeq \mathscr{M}^i$ induced by ι is morover filtration compatible, from which it follows that $\mathscr{M}^i$ is isomorphic to $\underline{\mathscr{M}}(M^i)$. As M^i is the Kisin module corresponds to $(T^i)^\vee$, Theorem 4.3 show that $\mathscr{M}^i$ is a strongly divisible lattice and $T_{\mathrm{st}}(\mathscr{M}) \simeq (T^i)^\vee$. By Theorem 3.8, we then have a natural isomorphism of Breuil-Kisin modules $\underline{M}(\mathscr{M}^i) \simeq M^i$ as desired. $\square$

6. Further directions

In this section, we discuss some questions and directions for further research.

6.1. Geometric interpretation of filtrations.

Let $\mathcal{X}$ be a smooth and proper scheme over $\mathscr{O}_K$, and let $(M, \mathrm{Fil}^i M, \varphi_{M,i})$ be the Breuil–Kisin module (in the sense of Definition 3.1) attached to the dual of the Galois lattice $T^i := H^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \mathbf{Z}_p)/\text{tors}$. For $p > 2$, when $i < p - 1$ and $H^j_{\text{cris}}(\mathcal{X}_k/W(k))$ is torsion free for $j = i, i+1$, our main result Theorem 5.3 provides the canonical "cohomological" interpretation

$$M \simeq \underline{M}(H^i_{\text{cris}}(\mathcal{X}_0/S)) := \varprojlim_{\varphi, n} \mathrm{Fil}^0(H^i_{\text{cris}}(\mathcal{X}_0/S) \otimes_S S_n[z_n^{-1}]).$$

However, our definition of the filtration on $\mathscr{M} := H^i_{\text{cris}}(\mathcal{X}_0/S)$—which plays a key role in the very definition of the Breuil–Kisin module $\underline{M}(\mathscr{M})$—is not as explicitly "geometric" as one might like. Indeed, put $V := T[1/p]$ and denote by $\widetilde{\mathscr{D}} := \mathscr{D}(V^\vee)$ the filtered (φ, N)-module over $S[1/p]$ attached to $D := D_{\text{cris}}(V^\vee)$ just above Definition 4.2. Using the Hyodo–Kato isomorphism $\mathscr{M}[1/p] \simeq \widetilde{\mathscr{D}}$ of φ-modules over $S[1/p]$, we equip $\mathscr{M}[1/p]$ with a filtration by "transport of structure", and have given the crystalline cohomology $\mathscr{M}$ of $\mathcal{X}_0$ the filtration $\mathrm{Fil}^i \mathscr{M} := \mathscr{M} \cap \mathrm{Fil}^i(\mathscr{M}[1/p])$; see §5.

We expect, however, that this filtration on $\mathscr{M}$ can be defined cohomologiclly as follows. For $m \geq 0$ set $E_m := \mathrm{Spec}(S/p^m S)$ and $\mathcal{Y}_m := \mathcal{X} \times_{\mathscr{O}_K} \mathscr{O}_K/p^m \mathscr{O}_K$, and let $\mathcal{J}_m$ be the sheaf of PD-ideals on the big crystalline site $\mathrm{Cris}(\mathcal{Y}_m/E_m)$ whose value on the object $(U \hookrightarrow T, \delta)$ is $\ker(\mathscr{O}_T \to \mathscr{O}_U)$. Writing $\mathcal{J}_m^{[i]}$ for the i-th divided power of $\mathcal{J}_m$, we expect that one has a canonical isomorphism

$$(6.1) \qquad \mathrm{Fil}^i(\mathscr{M}[1/p]) \simeq S[1/p] \otimes_S \varprojlim_m H^i((\mathcal{Y}_m/E_m)_{\text{cris}}, \mathcal{J}_m^{[i]}).$$

Unfortunately, we could not find a precise reference for this isomorphism in the literature. It would follow if one knew that the Hyodo–Kato isomorphism

$$H^i_{\text{cris}}(\mathcal{X}_0/S) \otimes_S K \simeq H^i_{\text{dR}}(\mathcal{X}_K/K)$$

carried the $S[1/p]$ submodule of $H^i_{\text{cris}}(\mathcal{X}_0/S) \otimes_S K$ given by the right side of (6.1) above isomorphically onto the K-subspace $\mathrm{Fil}^i H^i_{\text{dR}}(X_K/K)$ provided by the Hodge filtration. Presumably, this can be extracted from [17] or [26], and in any case we expect that a proof of such compatibility with filtrations can be given along the lines of the proof of [4, Proposition 13.9], using Berthelot's crystalline interpretation [3, Theorem 7.23] of the Hodge filtration. One can of course ask if the stronger, p-integral version of (6.1) holds as well, that is, whether or not (6.1) carries $\mathrm{Fil}^i \mathscr{M}$ isomorphically onto $\varprojlim_m H^i((\mathcal{Y}_m/E_m)_{\text{cris}}, \mathcal{J}_m^{[i]})$. If true, such an isomorphism would of course be the "best possible" cohomological interpretation of $\mathrm{Fil}^i \mathscr{M}$.

6.2. Other Frobenius Lifts and Wach Modules.

We expect that our main result can be generalized to give a cohomological description of the generalization of Breuil–Kisin modules constructed in [12], which include the Wach modules of Berger [1], [2], as well as the modules of Kisin–Ren [20]. More precisely, let $F \subseteq K$ be a subfield which is finite over $\mathbf{Q}_p$ with residue field k_F of cardinality $q = p^s$ and fixed uniformizer ϖ. Choose a power series $f(u) := a_1 u + a_2 u^2 + \cdots \in \mathscr{O}_F[[u]]$ with $f(u) \equiv u^q \mod \varpi$ and a uniformizer π_0 of K with minimal polynomial $E(u)$ over $F_0 := K_0 \cdot F$. Choose $\underline{\pi} := \{\pi_n\}_{n \geq 1}$ with $\pi_n \in \overline{K}$ satisfying $f(\pi_n) = \pi_{n-1}$ for $n \geq 1$. The resulting extension $K_{\underline{\pi}} := \bigcup_{n \geq 0} K(\pi_n)$ (called a *Frobenius iterate extension* in [11]) is an infinite and totally wildly ramified extension of K which in general need not be Galois, though in the special case that $v_\varpi(a_1) = 1$ and K is obtained from F by adjoining the roots of $f(u) = 0$, it is a Lubin–Tate extension of F.

Define $\mathfrak{S} := W[\![u]\!]$ and put $\mathfrak{S}_F = \mathscr{O}_F \otimes_{W(k_F)} \mathfrak{S}$. We equip $\mathfrak{S}_F$ with the (unique continuous) Frobenius endomorphism φ which acts on $W(k)$ by the q-power Witt-vector Frobenius, acts as the identity on $\mathscr{O}_F$, and sends u to $f(u)$. Define S_F to be the ϖ-adic completion of the $\mathscr{O}_F$-divided power envelope (in the sense of Faltings [15]) of the $\mathscr{O}_F$-algebra surjection $\mathfrak{S}_F \twoheadrightarrow \mathscr{O}_K$ sending u to π_0. There are evident analogues $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,r}$ and $\mathrm{Mod}_{S_F}^{\varphi,r}$ of the categories of Breuil–Kisin and Breuil modules in this setting, and the recent Ph. D. thesis of Henniges [16] shows that the canonical base change functor $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,r} \to \mathrm{Mod}_{S_F}^{\varphi,r}$ is an isomorphism when $p > 2$ and $r < q - 1$. We expect that the methods of the present paper can be adapted to provide an explicit quasi-inverse to this base change functor, along the lines of Definition 3.7 and Theorem 3.8. When one moreover has a theory of crystalline cohomology that produces Breuil modules over S_F, we further expect that Theorem 5.3 can be generalized, thereby giving a geometric description of the Breuil–Kisin modules constructed in [12] or in [20]. When F is unramified over $\mathbf{Q}_p$, so S_F is the usual completed PD-envelope of $\mathfrak{S} \twoheadrightarrow \mathscr{O}_K$, then the classical theory of crystalline cohomology already provides the necessary machinery to carry out this vision. For general F, one also has such a crystalline theory in the Barsotti–Tate setting $r = 1$ when K_π/F is Lubin–Tate, thanks to the work of Faltings [15].

Perhaps the simplest and most promising instance of the above framework is when $F = W(k)[1/p]$, $K = F(\mu_p)$ and $\varphi(u) = (1+u)^p - 1$. One may then choose $\underline{\pi}$ so that $K_{\underline{\pi}}$ is the cyclotomic extension of F. There is a natural action of $\Gamma := \mathrm{Gal}(K_{\underline{\pi}}/K)$ on $\mathfrak{S} = \mathfrak{S}_F$ given by $\gamma u := (1 + u)^{\chi(\gamma)} - 1$, which uniquely extends to $S = S_F$. We consider categories of modules $\mathrm{Mod}_{\mathscr{S}}^{\varphi,\Gamma,r}$ for $\mathscr{S} \in \{\mathfrak{S}, S\}$ whose objects are Breuil–Kisin or Breuil modules $(M, \mathrm{Fil}^r M, \varphi_{M,r})$ over $\mathscr{S}$ that have the additional structure of a semilinear Γ-action that is trivial on $M \otimes_{\mathscr{S}} W(k)$. The resulting category $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,\Gamma,r}$ of (φ, Γ)-modules over $\mathfrak{S}$ is equivalent to the category of *Wach modules*, as defined by [1] (though see [10, §4.5] for the claimed equivalences), which classify lattices in crystalline G_F-representations. We are confident that the main results of the present paper can be readily adapted to the above setting, thus giving a cohomological interpretation of Wach modules, at least in Hodge–Tate weights at most $p - 2$.

6.3. **Generalization to semistable schemes.** It is natural to ask to what extent the results of this paper can be generalized to the case of *semistable reduction*, that is, regular proper and flat schemes $\mathcal{X}$ over $\mathscr{O}_K$ with special fiber $\mathcal{X}_k$ that is a reduced normal corssings divisor on $\mathcal{X}$. It seems reasonable to guess that the analogue of Theorem 5.3 using log-crystalline cohomology continues to hold. In the case of low ramification $ei < p - 1$, it should be straightforward to prove that this is indeed the case using work of Caruso [13] (generalizing earlier work of Breuil [8] in the case $e = 1$), which provides the essential integral comparison isomorphisms needed to adapt the arguments of §5 to this setting. To get results without restriction on the ramification of K would require generalizing [4, Theorem 1.8] to the case of semistable reduction.

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