DIEUDONNÉ CRYSTALS AND WACH MODULES FOR
\textit{p}-DIVISIBLE GROUPS

BRYDEN CAIS AND EIKE LAU

Abstract. Let \( k \) be a perfect field of characteristic \( p > 2 \) and \( K \) an extension of \( F = \text{Frac} \ W(k) \) contained in some \( F(\mu_p^r) \). Using crystalline Dieudonné theory, we provide a classification of \( p \)-divisible groups over \( \mathcal{O}_K \) in terms of finite height (\( \varphi, \Gamma \))-modules over \( \mathcal{O}_K \). Although such a classification is a consequence of (a special case of) the theory of Kisin–Ren, our construction gives an independent proof and allows us to recover the Dieudonné crystal of a \( p \)-divisible group from the Wach module associated to its Tate module by Berger–Breuil or by Kisin–Ren.

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1. Introduction

Let $k$ be a perfect field of characteristic $p > 2$ and let $K$ be a finite extension of $F := \text{Frac} \, W(k)$ contained in $F_r := F(\mu_{p^r})$ for a fixed $r \geq 1$. As a special case of the theory of Kisin-Ren [31] one has the following description of the category $p\text{div}(\mathcal{O}_K)$ of $p$-divisible groups over the ring of integers of $K$ in terms of $(\varphi, \Gamma)$-modules.

Set $F_\infty := F(\mu_{p\infty})$ and let $\Gamma_K = \text{Gal}(F_\infty/K)$. The ring $\mathcal{S} := W(k)[u]$ is equipped with the cyclotomic Frobenius $\varphi(1 + u) = (1 + u)^p$ and with an action of $\Gamma_F$ by $\gamma(1 + u) = (1 + u)^{\chi(\gamma)}$ where $\chi$ is the $p$-adic cyclotomic character. We denote by $\text{BT}(\mathcal{S})_{\Gamma_K}$ the category of finite free $\varphi$-modules $\mathcal{M}$ over $\mathcal{S}$ such that $\mathcal{M}/\mathcal{M}^p$ is annihilated by $E := \varphi'(u)/\varphi'^{-1}(u)$, equipped with a semilinear action of $\Gamma_K$ that is trivial on $\mathcal{M}/u\mathcal{M}$.

**Theorem 1.** There is a contravariant equivalence of categories between $p\text{div}(\mathcal{O}_K)$ and $\text{BT}(\mathcal{S})_{\Gamma_K}$ sending $G$ to $\text{M}(G)$.

In the present article, we will derive this equivalence from crystalline Dieudonné theory, which gives an independent proof in this case and which also shows how to recover from $\text{M}(G)$ the Dieudonné crystal of $G$; see below. This can be important in applications, as for example in [13]. Along the way, various foundational results are stated in greater generality whenever this appeared reasonable. Explicitly, Theorem 1 is proved in Corollary 3.4.6; see Proposition 3.4.5 for a slightly more general statement. The equivalence is compatible with duality and with change of $r$ and $K$ and also allows to recover from $\text{M}(G)$ the Tate module of $G$; see Proposition 4.2.2.

As indicated above, using Kisin’s result [30] that every crystalline representation of the absolute Galois group $\mathcal{G}_K := \text{Gal}(\overline{F}/K)$ with Hodge-Tate weights 0 and 1 comes from a $p$-divisible group over $\mathcal{O}_K$, Theorem 1 is a consequence of the description of lattices in crystalline representations of $\mathcal{G}_K$ in terms of Wach modules by Wach [42], Colmez [17], Berger-Breuil [4], or in terms of the modules of Kisin-Ren [31]. In fact, the latter generalise a variant of the former to the case of a Lubin-Tate extension in place of the cyclotomic extension, and our modules $\text{M}(G)$ essentially coincide with those of [31, Corollary 3.3.8]; see §4.3.

Let us outline our line of thought, assuming that $K = F_r$ for simplicity. Let $S$ be the $p$-adic completion of the divided power envelope of the ideal $E\mathcal{S}$ of $\mathcal{G}$; note that $E$ is the minimal polynomial of a prime element of $\mathcal{O}_K$. We denote by $\text{Win}(\mathcal{S})_{\Gamma_K}$ the category of strongly divisible modules $M$ in the sense of Breuil [11], called windows by Zink [46], equipped with an action of $\Gamma_K$ that is trivial on $M \otimes_S W(k)$. There are two natural functors

$$p\text{div}(\mathcal{O}_K) \to \text{Win}(\mathcal{S})_{\Gamma_K} \leftrightarrow \text{BT}(\mathcal{S})_{\Gamma_K}$$

given by evaluating the Dieudonné crystal of a $p$-divisible group, and by a base change operation $\mathcal{M} \mapsto M = \mathcal{M} \otimes_{\mathcal{S}, \varphi} S$. The second functor is an equivalence by known results, and we will show that the same holds for the first functor.

Without the action of $\Gamma_K$, the construction makes sense for an arbitrary purely ramified extension $K$ of $F$ and for an arbitrary Frobenius $\varphi$ on $\mathcal{S}$. In this setting, the base change functor $\text{BT}(\mathcal{S}) \to \text{Win}(\mathcal{S})$ is always an equivalence by Caruso-Liu [15]. The functor $p\text{div}(\mathcal{O}_K) \to \text{Win}(\mathcal{S})$ is an equivalence only when the derivative
\( p^{-1}d\varphi \) is topologically nilpotent, which is for example satisfied when \( \varphi(u) = u^p \), but not when \( \varphi(1 + u) = (1 + u)^p \). At least for \( \varphi(u) = u^p \) this equivalence is due to Breuil [11]; the resulting equivalence between the categories \( \text{pdiv}(O_K) \) and \( \text{BT}(\mathcal{S}) \) was first proved by Kisin [30] using a different route.

In order to use Breuil’s result we have to switch between different choices of \( \varphi \). We consider the category \( \text{DF}(O_K) \) of Dieudonné crystals \((\mathcal{D},F,V)\) over \( \text{Spf}O_K \) equipped with an admissible Hodge filtration as in Definition 2.4.1 below. For every choice of \( \varphi \), this category is equivalent to the category \( \text{Win}(S)^\Gamma \) of windows with a connection. If the derivative \( p^{-1}d\varphi \) is topologically nilpotent, the forgetful functor \( \text{Win}(S)^\Gamma \rightarrow \text{Win}(S) \) is an equivalence. It follows that the crystalline Dieudonné module functor \( \text{pdiv}(O_K) \rightarrow \text{DF}(O_K) \) is an equivalence because both categories are equivalent to \( \text{Win}(S)^\Gamma \) when \( \varphi(u) = u^p \). This implies that for every \( \varphi \), the functor \( \text{pdiv}(O_K) \rightarrow \text{Win}(S)^\Gamma \) is an equivalence.

It remains to see that in the cyclotomic case \( K = F_r \), the category \( \text{Win}(S)^\Gamma_K \) is equivalent to \( \text{Win}(S)^\Gamma \). Indeed, the differential at the identity of an action of \( \Gamma_K \) gives a connection, and the exponential of the differential operator attached to a connection gives an action of \( \Gamma_K \). This correspondence is well-known in the context of \((\varphi,\Gamma)\)-modules over the Robba ring, see [2, §4.1], but additional care is required in the present situation as we must control the denominators of \( p \) which occur.

While the relation between \( \mathfrak{M}(G) \) and the Dieudonné crystal of \( G \) is clear from the construction, to recover the Tate module of \( G \) from \( \mathfrak{M}(G) \) we use a variant of Faltings’ integral comparison isomorphism for \( p \)-divisible groups [22, Theorem 7]. It is then straightforward to relate \( \mathfrak{M}(G) \) to the modules of Kisin-Ren [31] and Berger-Breuil [4].

Many of our arguments work naturally in greater generality than discussed above and may be of interest beyond the final application given here. For example, for the descent of windows from \( S \) to \( \mathcal{S} \) we need only that \( \mathcal{S} \) is a ring which is complete for the \((p,E)\)-adic topology where \((p,E)\) is a regular sequence; see Proposition 2.3.1. A similar result with additional hypotheses was recently obtained by Kim [29]. The relation between the categories of \( p \)-divisible groups, filtered modules, and Dieudonné crystals works essentially as explained above when \( O_K \) is replaced by a complete regular local ring with residue field \( k \); see §2.7. In the same spirit, we actually prove Theorem 1 in a version where \( O_K \) is replaced by a power series ring \( O_K[[t_1,\ldots,t_d]] \). This generalisation might not be the most reasonable, but it shows at least that some extension to higher dimensional bases is possible.

2. Frames, windows, and Dieudonné crystals

Let \( p \) be a prime. In this section, we recall and elaborate on the relation between Dieudonné crystals, Breuil modules, and Kisin modules associated to \( p \)-divisible groups following [5, 11, 19, 30, 29, 34, 46]. Most of this is well-known, but some aspects appear to be new. Technically we will use the notion of frames and windows, which we recall first.

2.1. Frames and windows. Frames and windows were introduced by Zink [46] and generalised in [33]. We use the definition of [33] with some minor modifications.
Definition 2.1.1. A frame \( S := (S, \text{Fil} S, R, \varphi, \varphi_1, \varpi) \) consists of a ring \( S \), an ideal \( \text{Fil} S \) of \( S \), the quotient ring \( R := S/\text{Fil} S \), a ring endomorphism \( \varphi : S \rightarrow S \) reducing to the \( p \)-power map modulo \( pS \), a \( \varphi \)-linear map \( \varphi_1 : \text{Fil} S \rightarrow S \), and an element \( \varpi \in S \) such that \( \varphi = \varpi \varphi_1 \) on \( \text{Fil} S \). We further require that \( \text{Fil} S + pS \subseteq \text{Rad}(S) \). We call \( S \) a lifting frame if in addition every finite projective \( R \)-module lifts to a finite projective \( S \)-module.

Remarks 2.1.2. In [33] the following surjectivity condition is also imposed: The image of \( \varphi_1 \) generates the unit ideal of \( S \). Then the element \( \varpi \) is determined by the rest of the data, namely if \( 1 = \sum a_i \varphi_1(b_i) \) then \( \varpi = \sum a_i \varphi(b_i) \). The surjectivity condition is satisfied in many examples, but is not necessary in the theory; cf. §11 of the arxiv version of [33]. It is not present in Zink’s notion of frames; see Definition 2.1.10 and Remark 2.1.11 below. If the surjectivity condition holds we will sometimes omit \( \varpi \) from the notation and write \( S = (S, \text{Fil} S, \varphi, \varphi_1) \) in order to be consistent with the literature.

In this paper, we will only consider lifting frames. All frames with \( R \) local are lifting frames (as projective \( R \)-modules of finite type are free). The condition \( \text{Fil} S + pS \subseteq \text{Rad}(S) \) is automatic if \( S \) is local and \( p \in R \) is a non-unit.

Definition 2.1.3. A window \( M := (M, \text{Fil} M, \Phi, \Phi_1) \) over a frame \( S \) consists of a projective \( S \)-module \( M \) of finite type, an \( S \)-submodule \( \text{Fil} M \subseteq M \) and \( \varphi \)-linear maps \( \Phi : M \rightarrow M \) and \( \Phi_1 : \text{Fil} M \rightarrow M \) such that:

1. There exists a decomposition of \( S \)-modules \( M = L \oplus N \) with \( \text{Fil} M = L \oplus (\text{Fil} S)N \).
2. If \( s \in \text{Fil} S \) and \( m \in M \) then \( \Phi_1(sm) = \varphi_1(s)\Phi(m) \).
3. If \( m \in \text{Fil} M \) then \( \Phi(m) = \varpi\Phi_1(m) \).
4. \( \Phi_1(\text{Fil} M) + \Phi(M) \) generates \( M \) as an \( S \)-module.

Naturally, a homomorphism of windows is an \( S \)-linear map that preserves the filtration and commutes with \( \Phi \) and with \( \Phi_1 \). We write \( \text{Win}(S) \) for the category of \( S \)-windows. A short sequence of windows is called exact if the sequences of \( M \)'s and of \( \text{Fil} M \)'s are both exact.

Remarks 2.1.4. If \( S \) satisfies the surjectivity condition of Remarks 2.1.2, then (3) follows from (2), condition (4) means that \( \Phi_1(\text{Fil} M) \) generates \( M \), and the map \( \Phi \) is determined by \( \Phi_1 \).

When \( S \) is a lifting frame, then (1) is equivalent to the simultaneous requirement that \( (\text{Fil} S)M \subseteq \text{Fil} M \) and that \( M/\text{Fil} M \) is projective as an \( R \)-module.

A decomposition as in (1) is called a normal decomposition. If \( (M, \text{Fil} M) \) and a normal decomposition \( M = L \oplus N \) are given, the set of pairs \( (\Phi, \Phi_1) \) which define a window \( M \) is in bijection with the set of \( \varphi \)-linear isomorphisms \( \Psi : L \oplus N \rightarrow M \) via \( \Psi(l + n) := \Phi_1(l) + \Phi(n) \).

Let us write \( F : \varphi^*M \rightarrow M \) for the linearization of \( \Phi \). There is a unique \( S \)-linear map \( V : M \rightarrow \varphi^*M \) with \( V(\Phi_1(m)) = 1 \otimes m \) for \( m \in \text{Fil} M \) and \( V(\Phi(m)) = \varpi \otimes m \) for \( m \in M \); here the last condition is automatic if \( S \) satisfies the surjectivity condition. The composition of \( F \) and \( V \) in either order is multiplication by \( \varpi \). See [34, Lemma 2.3].
Definition 2.1.5. A homomorphism of frames $\alpha : S \to S'$ is a homomorphism of rings $\alpha : S \to S'$ that intertwines $\varphi$ with $\varphi'$ and carries $\text{Fil} S$ into $\text{Fil} S'$ and which satisfies $\varphi' \alpha = c \cdot \alpha \varphi_1$ and $\alpha(\pi) = c \pi'$ for a unit $c \in S'$, which is part of the data. If we wish to specify $c$, we will say that $\alpha$ is a $c$-homomorphism. A strict homomorphism of frames is simply a 1-homomorphism.

Remark 2.1.6. If $S$ satisfies the surjectivity condition, then $c$ is uniquely determined by the relation $\varphi' \alpha = c \cdot \alpha \varphi$, and the relation $\alpha(\pi) = c \pi'$ is a consequence.

Let $\alpha : S \to S'$ be a $c$-homomorphism of frames. If $M$ and $M'$ are windows over $S$ and $S'$, then an $\alpha$-homomorphism of windows $f : M \to M'$ is a filtration-compatible homomorphism of $S$-modules $f : M \to M'$ that intertwines $\Phi$ with $\Phi'$ and satisfies $\Phi'_t f = c \cdot f \Phi_1$ on $\text{Fil} M$. Thus the morphisms in $\text{Win}(S)$ are the id$_S$-homomorphisms in this sense. There is a base change functor
\[
(2.1.1) \quad \alpha^* : \text{Win}(S) \longrightarrow \text{Win}(S')
\]
which is characterized by the universal property
\[
(2.1.2) \quad \text{Hom}_S(\alpha^* M, M') = \text{Hom}_\alpha(M, M')
\]
for $M \in \text{Win}(S)$ and $M' \in \text{Win}(S')$; here we write $\text{Hom}_\alpha(M, M')$ for the set of $\alpha$-homomorphisms. Explicitly, if $M = (M, \text{Fil} M, \Phi, \Phi_1)$ then the base change $\alpha^* M = (M', \text{Fil} M', \Phi', \Phi'_1)$ has $M' = S' \otimes_S M$ with $\text{Fil} M'$ the $S'$-submodule of $M'$ generated by $(\text{Fil} S') M'$ and the image of $\text{Fil} M$. The maps $\Phi'$ and $\Phi'_1$ are uniquely determined by the requirement that $M \to M'$ is an $\alpha$-homomorphism; in particular $\Phi'(s' \otimes m) = \varphi'(s') \otimes \Phi(m)$. See [34, §2.1] or [33, Lemma 2.9]. The frame homomorphism $\alpha$ is called crystalline if the base change functor (2.1.1) is an equivalence of categories.

Definition 2.1.7. The dual of an $S$-window $M$ is $M^t := (M^t, \text{Fil} M^t, \Phi^t, \Phi'_1)$ where:

1. $M^t := \text{Hom}_S(M, S)$ is the $S$-linear dual of $M$,
2. $\text{Fil} M^t := \{f \in M^t : f(\text{Fil} M) \subseteq \text{Fil} S\}$,
3. The maps $\Phi^t : M^t \to M^t$ and $\Phi'_1 : \text{Fil} M^t \to M^t$ are determined by the relations
\[
\begin{align*}
\Phi'_1(f)(\Phi_1(m)) &= \varphi_1(f(m)) & \text{for } f \in \text{Fil} M^t \text{ and } m \in \text{Fil} M, \\
\Phi'_1(f)(\Phi(m)) &= \varphi(f(m)) & \text{for } f \in \text{Fil} M^t \text{ and } m \in \text{Fil} M, \\
\Phi^t(f)(\Phi_1(m)) &= \varphi(f(m)) & \text{for } f \in M^t \text{ and } m \in \text{Fil} M, \\
\Phi^t(f)(\Phi(m)) &= \pi \varphi(f(m)) & \text{for } f \in M^t \text{ and } m \in M.
\end{align*}
\]

Remarks 2.1.8. If the surjectivity condition of Remark 2.1.2 holds for $S$, then the first of the relations in (3) implies the others.

The existence of the pair of maps $(\Phi^t, \Phi'_1)$ requires an argument using normal representations; see [34, §2.1] or [33, §2].

There is a canonical “double duality” isomorphism $M'' \simeq M$ in $\text{Win}(S)$.

If $F : \varphi^* M \to M$ and $V : M \to \varphi^* M$ are the $S$-linear maps defined in Remark 2.1.4, then the corresponding maps for $M^t$ are the $S$-linear duals of $V$ and $F$, respectively.
The formation of duals commutes with base change along strict homomorphisms of frames, and also along a \( c \)-homomorphism \( S \to S' \) provided a unit \( y \in S' \) with \( c = y/\varphi'(y) \) is given: there is always a canonical isomorphism \( (\alpha^* M)^t \cong \alpha^* (M^t)_{c^{-1}} \) where the subscript means that \( \Phi \) and \( \Phi_1 \) are multiplied by \( c^{-1} \), and multiplication by \( y \) then gives an isomorphism \( y: \alpha^* (M^t)_{c^{-1}} \to \alpha^* (M^t) \); see [33, Lemma 2.14].

**Example 2.1.9.** For every frame \( S \) we have the windows \( S = (S, \text{Fil} S, \varphi, \varphi_1) \) and its dual \( S^! = (S, S, \varpi \varphi, \varphi_1) \).

Let us recall the frames of [46], which we call PD-frames here:

**Definition 2.1.10.** Let \( R \) be a \( p \)-adically complete ring. A PD-frame for \( R \) is a surjective ring homomorphism \( S \to R \) together with a Frobenius lift \( \varphi : S \to S \) where \( S \) is \( p \)-adically complete without \( p \)-torsion such that the kernel of \( S \to R \) is a PD-ideal.

**Remark 2.1.11.** A PD-frame for \( R \) satisfies \( \varphi(S) \subseteq pS \) and therefore extends uniquely to a frame \( S = (S, \text{Fil} S, R, \varphi, \varphi_1, p) \) in the sense of Definition 2.1.1, i.e. \( \varphi_1(a) = p^{-1} \varphi(a) \). Moreover \( S \) is a lifting frame because the kernel of \( S/pS \to R/pR \) is a nilideal due to the divided powers. We will also call the collection \( S \) a PD-frame.

Our second main example of frames is the following:

**Example 2.1.12.** Assume that \( \mathcal{S} \) is a ring with a fixed non-zero divisor \( E \in \mathcal{S} \) such that \( p, E \in \text{Rad}(\mathcal{S}) \), and \( \varphi : \mathcal{S} \to \mathcal{S} \) is an endomorphism that induces the \( p \)-power map on \( \mathcal{S}/p\mathcal{S} \). We can form the frame

\[
\mathcal{S} = (\mathcal{S}, \text{Fil} \mathcal{S}, R, \varphi, \varphi_1)
\]

with \( \text{Fil} \mathcal{S} = E \mathcal{S} \) and \( R = \mathcal{S}/E \mathcal{S} \) and \( \varphi_1(Ex) = \varphi(x) \) for \( x \in \mathcal{S} \). The element \( \varpi \) with \( \varphi = \varpi \varphi_1 \) on \( \text{Fil} \mathcal{S} \) is uniquely determined by \( \varpi = \varphi(E) \) and is omitted from the notation; cf. Remark 2.1.2.

**Definition 2.1.13.** A Barsotti-Tate (BT) module over \( \mathcal{S} \) is a pair \((M, \varphi)\) where \( M \) is a finite projective \( \mathcal{S}\)-module and \( \varphi : M \to M \) is a \( \varphi \)-linear map such that the cokernel of the associated linear map \( 1 \otimes \varphi : \varphi^* M \to M \) is annihilated by \( E \) and projective as an \( R \)-module. The category of BT modules over \( \mathcal{S} \) is denoted by \( \text{BT}(\mathcal{S}) \).

If \( R \) is a regular ring, the last projectivity condition is automatic.

**Remark 2.1.14.** One verifies that the category \( \text{BT}(\mathcal{S}) \) can be equipped with the following duality operation: The dual of \((M, \varphi)\) is \((M^t, \varphi^t)\) where \( \varphi^t \) is determined by \( \varphi^t(f)(\varphi(m)) = E \varphi(f(m)) \) for \( f \in M^t \) and \( m \in M \).

**Lemma 2.1.15.** If all finite projective \( R \)-modules lift to finite projective \( \mathcal{S} \)-modules, there is an equivalence of categories \( \text{Win}(\mathcal{S}) \to \text{BT}(\mathcal{S}) \) given by

\[
M = (M, \text{Fil} M, \Phi, \Phi_1) \mapsto (\text{Fil} M, E\Phi_1).
\]

The equivalence preserves exactness and duality.

In many examples the hypothesis of Lemma 2.1.15 is satisfied because \( \mathcal{S} \) is local or \( E \)-adically complete.
Proof. This is standard. Let $M$ be an $\mathcal{G}$-window. Then the $\mathcal{G}$-module $\mathcal{M} := \text{Fil} M$ is projective, and the linearization of $\Phi_1 : \mathcal{M} \rightarrow M$ is an isomorphism $\varphi^{\ast} \mathcal{M} \cong M$. The inclusion $\mathcal{M} \rightarrow M$ composed with the inverse of this isomorphism defines a linear map $\psi : \mathcal{M} \rightarrow \varphi^{\ast} \mathcal{M}$, and there is a unique linear map $\tilde{\varphi} : \varphi^{\ast} \mathcal{M} \rightarrow \mathcal{M}$ with $\tilde{\varphi} \psi = E \cdot \text{id}$. The cokernel of $\varphi$ is isomorphic to $\text{Fil} M / E M$, which is projective over $R$. Let $\varphi(x) = \tilde{\varphi}(1 \otimes x)$ for $x \in \mathcal{M}$. Then $(\mathcal{M}, \varphi) \in \text{BT}(\mathcal{G})$, and one verifies that $\varphi = E \Phi_1$.

Conversely, for $(\mathcal{M}, \varphi) \in \text{BT}(\mathcal{G})$ there is a unique linear map $\psi : \mathcal{M} \rightarrow \varphi^{\ast} \mathcal{M}$ with $\tilde{\varphi} \psi = E \cdot \text{id}$, and we get a window $\tilde{M} = (M, \text{Fil} M, \Phi, \Phi_1)$ by setting $M := \varphi^{\ast} \mathcal{M}$ and $\text{Fil} M := \psi(\mathcal{M})$ and $\Phi_1(\psi(x)) := 1 \otimes x$ for $x \in \mathcal{M}$ and $\Phi(x) := 1 \otimes \varphi(x)$ for $x \in M$.

Example 2.1.16. The window $\mathcal{G} = (\mathcal{G}, E \mathcal{G}, \varphi, \varphi')$ corresponds to the BT module $(\mathcal{G}, \varphi)$, and its dual $\mathcal{G}' = (\mathcal{G}, \mathcal{G}, (E)\varphi, \varphi)$ corresponds to $(\mathcal{G}, E \varphi)$.

2.2. Lifting windows modulo powers of $p$. As a preparation for Proposition 2.3.1 below, we consider the following situation. Let

$$S' = (S', \text{Fil} S', R', \varphi', \varphi_1', \varpi') \xrightarrow{\lambda} S = (S, \text{Fil} S, R, \varphi, \varphi_1, \varpi)$$

be a frame homomorphism such that the rings $S', R', S, R$ are $p$-adically complete and $\mathbb{Z}_p$-flat. For each $n \geq 1$ we obtain frames $S_n = S \otimes \mathbb{Z}/p^n \mathbb{Z}$ and $S_n' = S' \otimes \mathbb{Z}/p^n \mathbb{Z}$, the tensor product taken componentwise, and a frame homomorphism $\lambda_n : S_n' \rightarrow S_n$. Recall that a frame homomorphism is called crystalline if it induces an equivalence of the associated window categories.

Proposition 2.2.1. If $\lambda_1$ is crystalline, then all $\lambda_n$ and $\lambda$ are crystalline.

Proof. For a fixed $S_n$-window $M_n = (M, \text{Fil} M, \Phi, \Phi_1)$ we write $\mathcal{L}(M_n)$ the category of lifts of $M_n$ to an $S_{n+1}$-window $\tilde{M}_n = (\tilde{M}, \text{Fil} \tilde{M}, \tilde{\Phi}, \tilde{\Phi}_1)$; the morphisms in this category are isomorphisms that induce the identity of $M_n$. Let $M = M \otimes \mathbb{Z}/p \mathbb{Z} = (\tilde{M}, \text{Fil} \tilde{M}, \tilde{\Phi}, \tilde{\Phi}_1)$. We claim that there is an equivalence of categories

$$(2.2.1) \quad \mathcal{L}(M_n) \cong \text{Ext}^1_{\mathcal{S}_1}(\tilde{M}, \tilde{M}),$$

depending on the choice of a base point in $\mathcal{L}(M_n)$. All lifts to $S_{n+1}$ of the pair $(M, \text{Fil} M)$ are isomorphic, and we choose one of them $(\tilde{M}, \text{Fil} \tilde{M})$. Let $D^1$ be the set of pairs $(G, G_1)$ of $\varphi$-linear maps $G : \tilde{M} \rightarrow \tilde{M}$ and $G_1 : \text{Fil} \tilde{M} \rightarrow \tilde{M}$ that satisfy the relations $G_1(ax) = \varphi_1(a)G(x)$ for $a \in \text{Fil} S_1$ and $x \in \tilde{M}$ and $G(x) = \varpi G_1(x)$ for $x \in \text{Fil} \tilde{M}$. The set of pairs $(\Phi, \Phi_1)$ which complete the lift $\tilde{M}$ is a principally homogeneous set under the abelian group $D^1$, where $(G, G_1)$ acts on $(\tilde{\Phi}, \tilde{\Phi}_1)$ by adding $(p^n G, p^n G_1)$. The group of automorphisms of the chosen lift $(\tilde{M}, \text{Fil} \tilde{M})$ is isomorphic to $D^0 := \text{End}(\tilde{M}, \text{Fil} \tilde{M})$ by sending $a \in D_0$ to the automorphism $1 + p^n a$, and the action of these automorphisms on the pairs $(\tilde{\Phi}, \tilde{\Phi}_1)$ is given by

$$d : D^0 \rightarrow D^1, \quad d(a) = (\tilde{\Phi} a - a \tilde{\Phi}, \tilde{\Phi}_1 a - a \tilde{\Phi}_1).$$

It follows that the category $\mathcal{L}(M_n)$ is equivalent to the quotient groupoid $[D_1/D_0]$. Similary, for an extension of $S_n$-windows $0 \rightarrow \tilde{M} \rightarrow N \rightarrow \tilde{M} \rightarrow 0$, the underlying
pair of modules \((N, \Fil N)\) is isomorphic to \((\overline{M} \oplus M, \Fil \overline{M} \oplus \Fil M)\). The operators \(\Phi_N\) and \(\Phi_{1,N}\) of \(N\) are then given by block matrices \(\begin{pmatrix} \overline{G} & G \\ 0 & \overline{G} \end{pmatrix}\) and \(\begin{pmatrix} \overline{G}_1 & G_1 \\ 0 & \overline{G}_1 \end{pmatrix}\) for some \((G, G_1) \in D_1\), the automorphism group of \((N, \Fil N)\) is isomorphic to \(D^0\) by sending \(\alpha \in D_0\) to the automorphism \((\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})\), and the action of automorphisms on pairs \((\Phi_N, \Phi_{1,N})\) is given by \(d\) as above. The equivalence (2.2.2) follows. For an \(S_n\)-window \(\overline{M}_n\) we get a similar equivalence
\[
\mathcal{L}(\overline{M}_n') \cong \mathcal{E}xt^1_{\mathfrak{S}_1}(\overline{M'}, \overline{M'}). 
\]
We apply this for \(\overline{M}_n = \lambda_n(\overline{M}_n')\) and deduce that since \(\lambda_1\) is assumed to be crystalline, the functor
\[
(2.2.2) \quad \mathcal{L}(\overline{M}_n') \to \mathcal{L}(\overline{M}_n)
\]
induced by \(\lambda_{n+1}\) is an equivalence of categories.

Now to prove the proposition it suffices to show that if \(\lambda_n\) is crystalline, then so is \(\lambda_{n+1}\). Assume that the functor \(\lambda_n^*\) is an equivalence.

a) The functor \(\lambda^*_{n+1}\) is faithful: Let \(\overline{M}'_{n+1}\) and \(\overline{N}'_{n+1}\) be two \(S_{n+1}\)-windows, let \(\overline{M}'_n\) and \(\overline{N}'_n\) be their reductions modulo \(p^n\), and let \(\overline{M}_n\) and \(\overline{N}_n\) be the images under \(\lambda^*_n\). Assume that \(\alpha : \overline{M}'_{n+1} \to \overline{N}'_{n+1}\) is a homomorphism with \(\lambda^*_{n+1}(\alpha_{n+1}) = 0\). Since the functor \(\lambda^*_n\) is faithful we have \(\alpha = 0\) for a homomorphism \(\beta : \overline{M}'_n \to \overline{N}'_n\). Since the functor \(\lambda^*_n\) is faithful we have \(\beta = 0\), thus \(\alpha = 0\).

b) The functor \(\lambda^*_{n+1}\) is full: Since a homomorphism \(\alpha : M \to N\) can be encoded by the automorphism \((\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})\) of \(M \oplus N\), it suffices to show that \(\lambda^*_{n+1}\) is full on isomorphisms. We use the notation of a). Let \(\alpha_{n+1} : \overline{M}'_{n+1} \to \overline{N}'_{n+1}\) be a given isomorphism, and let \(\overline{\alpha}_n : \overline{M}'_n \to \overline{N}'_n\) be the unique isomorphism that lifts the reduction \(\alpha_n : \overline{M}_n \to \overline{N}_n\). Since \(\alpha_n\) can be lifted to an isomorphism of pairs \((\overline{M}'_{n+1}, \Fil \overline{M}'_{n+1}) \cong (\overline{N}'_{n+1}, \Fil \overline{N}'_{n+1})\), we can assume that \(\overline{M}'_n = \overline{N}'_n\) and that \(\alpha_n = \id\). Then the equivalence (2.2.2) implies that \(\alpha\) comes from an isomorphism \(\overline{M}'_{n+1} \cong \overline{N}'_{n+1}\).

c) The functor \(\lambda^*_{n+1}\) is surjective on isomorphism classes: This is clear by the equivalence (2.2.2).

\(\square\)

### 2.3. Descent of windows from \(S\) to \(\mathfrak{S}\)

Let \(\mathfrak{S}\) be a ring with a fixed element \(E \in \mathfrak{S}\) such that \((p, E)\) is a regular sequence in \(\mathfrak{S}\). We assume that \(\mathfrak{S}\) is complete for the \((p, E)\)-adic topology, or equivalently that \(\mathfrak{S}\) is \(p\)-adically complete and \(\mathfrak{S}_1 = \mathfrak{S}/p\mathfrak{S}\) is \(E\)-adically complete. Then \(\mathfrak{S}\) is also \(E\)-adically complete without \(E\)-torsion, and \(R = \mathfrak{S}/E\mathfrak{S}\) is \(p\)-adically complete and \(\mathbb{Z}_p\)-flat. Let \(\varphi : \mathfrak{S} \to \mathfrak{S}\) be a lift of Frobenius. As in Example 2.1.12 we form the frame \(\mathfrak{S} = (\mathfrak{S}, \Fil \mathfrak{S}, R, \varphi, \varphi')\).

Let \(S\) be the \(p\)-adic completion of the divided power envelope of the ideal \(E\mathfrak{S} \subseteq \mathfrak{S}\), let \(\Fil S\) be the kernel of \(S \to R\), and let \(\varphi : S \to S\) be the natural extension of \(\varphi\). The ring \(S\) is \(\mathbb{Z}_p\)-flat (see Lemma 2.6.1 below) and is a PD-frame for \(R\) in the sense of Definition 2.1.10. Let
\[
\mathfrak{S} = (S, \Fil S, R, \varphi, \varphi_1, p)
\]
be the corresponding frame, i.e. \(\varphi_1(a) = p^{-1}\varphi(a)\) for \(a \in \Fil S\). We assume that \(c := \varphi_1(E)\) is a unit of \(S\). Then the natural homomorphism \(\mathfrak{S} \to S\) is a \(c\)-homomorphism of frames \(\lambda : \mathfrak{S} \to S\). Let \(\lambda_n : \mathfrak{S}_n \to S_n\) be its reduction mod \(p^n\) as in §2.2.
Proposition 2.3.1. If \( p \geq 3 \) then the frame homomorphisms \( \lambda : \mathfrak{S} \to S \) and \( \lambda_n : \mathfrak{S}_n \to S_n \) for \( n \geq 1 \) are all crystalline.

This generalises known results. When \( \mathfrak{S} = W(k)[u] \) for a perfect field \( k \) of characteristic \( p \) and when \( E \) is an Eisenstein polynomial so that \( R \) is the ring of integers of a \( p \)-adic local field, the equivalence between \( \mathfrak{S} \)-windows and \( S \)-windows follows from [30, 2.2.7, A.6] when \( \varphi(u) = u^p \) and, more directly, from [15, 2.2.1] for general \( \varphi \). Using the method of [15], this equivalence is extended in [29, Prop. 6.3] to the case \( \mathfrak{S} = R_0[u] \) for a \( p \)-adically complete ring \( R_0 \) where \( E \in \mathfrak{S} \) is a generalisation of an Eisenstein polynomial.

Proof. By Proposition 2.2.1 it suffices to show that the frame homomorphism \( \mathfrak{S}_1 \to S_1 \) is crystalline. We prove this using a variant of [9, Pr. 2.2.2.1] and [10, Th. 4.1.1].

Let \( \mathfrak{S}_0 = \mathfrak{S}/(p, E^p) = \mathfrak{S}_1/(E^p) \). For \( x \in \mathfrak{S}_1 \) we have
\[
\varphi'_1(xE^p) = \varphi(x)\varphi(E)^{p-1} = x^p E^p(p-1)
\]
in \( \mathfrak{S}_1 \). Thus \( \varphi'_1 \) preserves the ideal \( J = E^p \mathfrak{S}_1 \), and (since \( p \geq 3 \)) the restriction \( \varphi'_1 : J \to J \) is topologically nilpotent. It follows that (also for \( p = 2 \)) we have a frame
\[
\mathfrak{S}_0 = (\mathfrak{S}_0, \Fil \mathfrak{S}_0, R_1, \varphi, \varphi'_1)
\]
such that the projection \( \pi : \mathfrak{S}_1 \to \mathfrak{S}_0 \) is a strict frame homomorphism \( \pi : \mathfrak{S}_1 \to \mathfrak{S}_0 \), and for \( p \geq 3 \) this frame homomorphism is crystalline by the general deformation result [33, Theorem 3.2].

We can endow the ideal \( \Fil \mathfrak{S}_0 = E\mathfrak{S}_1/E^p \mathfrak{S}_1 \) with the trivial divided powers \( \gamma \) determined by \( \gamma_p(E) = 0 \). The universal property of \( S_1 \) gives an extension of \( \pi \) to a divided power homomorphism \( \pi' : S_1 \to \mathfrak{S}_0 \), which maps \( \Fil \mathfrak{S}_1 \) to zero. Now the identity of \( \mathfrak{S}_0 \) factors into
\[
\mathfrak{S}_0 = \mathfrak{S}_1/E^p \mathfrak{S}_1 \to S_1/\Fil \mathfrak{S}_1 \to \mathfrak{S}_0.
\]
Here the first arrow is easily seen to be surjective, thus both arrows are bijective. In \( S \) we have the relation
\[
\varphi(E^{n}/n!) = e^n p^{n-1}/n! = \text{unit} \cdot p^{n-1-v_p(n!)}
\]
and \( v_p(n!) \leq (n-1)/(p-1) \). As \( p \geq 3 \), the exponent \( n - 1 - v_p(n!) \) is positive for \( n \geq 2 \). Thus \( \varphi_1 : \Fil \mathfrak{S}_1 \to S_1 \) is zero, and the map \( \varphi_1 : \Fil \mathfrak{S}_1 \to S_1 \) factors over \( \Fil \mathfrak{S}_0 \to \mathfrak{S}_1 \) and in particular induces \( \varphi_1 : \Fil \mathfrak{S}_0 \to \mathfrak{S}_0 \). Using that \( \lambda \) is a \( c \)-homomorphism of frames, we find that \( \varphi_1 = c\varphi'_1 \). Thus \( \varphi' \) is a \( c^{-1} \)-homomorphism of frames \( \pi' : S_1 \to \mathfrak{S}_0 \), which is crystalline by [33, Theorem 3.2] again. \( \square \)

2.4. Dieudonné crystals and the Hodge filtration. Let \( T \) be a scheme on which \( p \) is nilpotent. We denote by \( \mathcal{D}(T) \) the category of Dieudonné crystals over \( T \). Let us recall the definition:

Set \( \Sigma := \Spec \mathbb{Z}_p \), endowed with the structure of a PD-scheme via the canonical divided powers on the ideal \( p\mathbb{Z}_p \), and let \( \text{Cris}(T/\Sigma) \) be the big fpf crystalline site as in [5]. Then an object of \( \mathcal{D}(T) \) is a triple \( (\mathcal{E}, F, V) \) where \( \mathcal{E} \) is a crystal of finite locally free \( \mathcal{O}_{T/\Sigma} \)-modules on \( \text{Cris}(T/\Sigma) \) and where \( F : \varphi^* \mathcal{F}_0 \to \mathcal{F}_0 \) and \( V : \mathcal{V}_0 \to \varphi^* \mathcal{V}_0 \) are homomorphisms of crystals satisfying \( FV = p \) and \( VF = p \).
here $\mathcal{D}_0$ denotes the pullback of $\mathcal{D}$ to $T_0 = T \times \text{Spec } \mathbb{F}_p$, and $\varphi$ is the Frobenius endomorphism of $T_0$. The dual of $(\mathcal{D}, F, V)$ is $(\mathcal{D}^t, V^t, F^t)$.

Let $\text{pddiv}(T)$ denote the category of $p$-divisible groups over $T$. We have a contravariant functor

$$\mathbf{D} : \text{pddiv}(T) \to \mathbb{D}(T), \quad \mathbf{D}(G) = (\mathbf{D}(G), F, V)$$

where $\mathbf{D}(G)$ is the crystalline Dieudonné module of $G$ defined in [38] or [5], and where $F$ and $V$ are induced by the Frobenius and Verschiebung homomorphism of $G_{T_0}$. The functor $\mathbf{D}$ is exact by [5, Pr. 4.3.1 and Th. 3.3.3] and compatible with duality by [5, Sec. 5.3].

We want to take into account the Hodge filtration of a $p$-divisible group. Let $(\mathcal{D}, F, V)$ be a Dieudonné crystal over $T$. For each object $(U \to Z)$ of $\text{Cris}(T_0/\Sigma_0)$ we have an exact sequence of locally free $\mathcal{O}_Z$-modules

$$\mathcal{D}_{Z/U} \xrightarrow{V_{Z/U}} \varphi^*(\mathcal{D}_{Z/U}) \xrightarrow{F_{Z/U}} \mathcal{D}_{Z/U} \xrightarrow{V_{Z/U}} \varphi^*(\mathcal{D}_{Z/U})$$

where $\varphi$ denotes the Frobenius endomorphism of $Z$, and $\text{im}(V_{Z/U})$ and $\text{im}(F_{Z/U})$ are locally free as well. Indeed, (2.4.2) is a complex of locally free modules whose base change to perfect fields is exact, which implies the assertion. Moreover, the Frobenius endomorphism $\varphi : Z \to Z$ factors into $\varphi : Z \to U$.

**Definition 2.4.1.** We denote by $\mathbf{DF}(T)$ the category of filtered Dieudonné crystals over $T$ whose objects are quadruples $(\mathcal{D}, F, V, \text{Fil } \mathcal{D})$ where $(\mathcal{D}, F, V)$ is a Dieudonné crystal over $T$ and where $\text{Fil } \mathcal{D}_{T} \subseteq \mathcal{D}_T$ is an $\mathcal{O}_T$-submodule which is locally a direct summand, such that for each object $(U \to Z)$ of $\text{Cris}(T_0/\Sigma_0)$, we have

$$\varphi^*((\text{Fil } \mathcal{D}_T)_U) = \text{im}(V_{Z/U})$$

inside the locally free $\mathcal{O}_Z$-module $\varphi^*\mathcal{D}_U = \varphi^*\mathcal{D}_{Z/U}$. A short sequence of filtered Dieudonné crystals is called exact if the underlying sequences of $\mathcal{O}_T$-$\Sigma$-modules $\mathcal{D}$ and of $\mathcal{O}_T$-modules $\text{Fil } \mathcal{D}_T$ are exact. The dual of $(\mathcal{D}, F, V, \text{Fil } \mathcal{D}_T)$ is defined as $(\mathcal{D}^t, V^t, F^t, (\text{Fil } \mathcal{D}_T)^t)$.

**Remark 2.4.2.** The locally direct summands of $\mathcal{D}_T$ that satisfy (2.4.3) are precisely the [admissible] filtrations considered in [27, p. 113].

For a $p$-divisible group $G$ over $T$ we have a natural exact sequence of locally free $\mathcal{O}_T$-modules, called the Hodge filtration of $G$,

$$0 \to \omega_G \to \mathbf{D}(G)_T \to \mathcal{L}ie(G^\vee) \to 0.$$

**Proposition 2.4.3.** The functor $\mathbf{D}$ of (2.4.1) extends to a contravariant functor

$$\mathbf{D} : \text{pddiv}(T) \to \mathbf{DF}(T), \quad \mathbf{D}(G) = (\mathbf{D}(G), F, V, \text{Fil } \mathbf{D}(G)_T)$$

where $\text{Fil } \mathbf{D}(G)_T$ is the image of $\omega_G$. This functor is again exact and compatible with duality.

**Proof.** We have to verify that (2.4.3) holds with this definition of $\text{Fil } \mathbf{D}(G)_T$. By [5, Pr. 4.3.10] the assertion holds when $U = Z$. Since $G_U$ can locally in $U$ be lifted to a $p$-divisible group over $Z$, the assertion follows in general. The extended functor $\mathbf{D}$ is
exact because the functor \( G \mapsto \mathcal{L}ie(G) \) is exact; cf. [39, Th. 3.3.13]. The extended functor \( D \) is compatible with duality by [5, Pr. 5.3.6].

2.4.4. If \( T \) is a formal scheme over \( \text{Spf } \mathbb{Z}_p \) with ideal of definition \( \mathcal{I} \subseteq \mathcal{O}_T \), we define a (filtered) Dieudonné crystal over \( T \) to be a compatible system of such objects over the schemes \( T_n = V(\mathcal{I}^n) \) for \( n \geq 1 \). The preceding discussion extends to this case.

2.5. Dieudonné crystals and windows. Let \( \mathcal{S} = (S, \text{Fil } S, R, \varphi, \varphi_1, p) \) be a PD-frame for a \( p \)-adically complete ring \( R \) in the sense of Remark 2.1.11. If a Dieudonné crystal \( (D, F, V) \) over \( \text{Spf } R \) together with a locally direct summand \( \text{Fil } D_R \subseteq D_R \) is given, one can try to define a window \( M \) over \( \mathcal{S} \) as follows. Set \( M = D_S \), let \( \text{Fil } M \subseteq M \) be the inverse image of \( \text{Fil } D_R \subseteq D_R = M/(\text{Fil } S)M \), and let \( \Phi : M \rightarrow M \) be the \( \varphi \)-linear map induced by \( F \).

Lemma 2.5.1. The following conditions are equivalent.

1. We have \( \Phi(\text{Fil } M) \subseteq pM \), and \( \Phi(M) + p^{-1}\Phi(\text{Fil } M) \) generates \( M \) over \( S \).
2. The equality (2.4.3) holds for \( U = \text{Spec } R/pR \) and \( Z = \text{Spec } S/pS \).

Condition (1) means that \( M = (M, \text{Fil } M, \Phi, \Phi_1 = p^{-1}\Phi) \) is a well-defined \( S \)-window. Thus we get:

Proposition 2.5.2. There is an exact functor

\[
\text{DF}(\text{Spf } R) \rightarrow \text{Win}(\mathcal{S}), \quad D \mapsto M
\]

that is compatible with duality. \( \square \)

Proof of Lemma 2.5.1. We may assume that \( p \) is nilpotent in \( R \). Since the Frobenius endomorphism of \( S/pS \) induces a homomorphism \( \varphi : R/pR \rightarrow S/pS \), the \( \varphi \)-linear map \( \Phi : M \rightarrow M \) induces a \( \varphi \)-linear map

\[
\Phi : R/pR \otimes_S M \rightarrow M/pM.
\]

The condition \( \Phi(\text{Fil } M) \subseteq pM \) is equivalent to \( \Phi(\text{Fil } D_R/pR) = 0 \), which translates into

(2.5.1) \[
\varphi^* (\text{Fil } D_R/pR) \subseteq \text{im}(V_{Z/U})
\]

inside \( \varphi^* D_U = \varphi^* D_{Z/U} \) for \( U = \text{Spec } R/pR \) and \( Z = \text{Spec } S/pS \); this is one inclusion of the equality (2.4.3). Assume that (2.5.1) holds so that we can define \( \Phi_1 := p^{-1}\Phi : \text{Fil } M \rightarrow M \). It remains to show that (2.5.1) is an equality if and only if \( \Phi(M) + \Phi_1(\text{Fil } M) \) generates \( M \) over \( S \). This is easily verified if \( R = k \) is a perfect field and \( S = W(k) \). The general case is reduced to this case as follows.

We choose a normal decomposition \( M = L \oplus N \) such that \( \text{Fil } M = L \oplus (\text{Fil } S)N \). The module \( M \) is generated by \( \Phi(M) + \Phi_1(M) \) if and only if the \( \varphi \)-linear map of \( S \)-modules \( \Psi : L \oplus N \rightarrow M \) defined by \( \Phi_1 \) on \( L \) and by \( \Phi \) on \( N \) is a \( \varphi \)-linear isomorphism. Every homomorphism \( R \rightarrow k \) with a perfect field \( k \) extends uniquely to a homomorphism of frames \( \mathcal{S} \rightarrow W(k) \), so this condition can be checked over perfect fields. Similarly, since both sides of (2.5.1) are direct summands of \( M/pM \), we have equality if and only if equality holds after every base change to a perfect field. \( \square \)
2.6. Dieudonné crystals and windows with connection. Assume that \( A \to R = A/J \) is a surjective homomorphism of \( p \)-adically complete rings where \( A \) is \( \mathbb{Z}_p \)-flat. Let \( S \) be the \( p \)-adic completion of the divided power envelope of \( J \) in \( A \) with respect to \( \Sigma \); see §2.4. Then \( S \) depends only on the ideal \( J + pA \). Let \( \text{Fil} S \) be the kernel of the natural homomorphism \( S \to R \).

**Lemma 2.6.1.** If the kernel of \( A/pA \to R/pR \) is generated by a regular sequence locally in \( \text{Spec} A/pA \), then the ring \( S \) is \( \mathbb{Z}_p \)-flat.

This should be standard, but we include a proof for completeness.

**Proof.** We may assume that \( J_0 = \ker(A/pA \to R/pR) \) is generated by a regular sequence \( \tilde{t}_1, \ldots, \tilde{t}_r \). Let \( t_i \in A \) be an inverse image of \( \tilde{t}_i \), and let \( J' = (t_1, \ldots, t_r) \) as an ideal of \( A \). Then \( t_1, \ldots, t_r \) is a regular sequence, and \( R' = A/J' \) is a \( \mathbb{Z}_p \)-flat \( p \)-adically complete ring. Without changing \( S \) we can assume that \( J = J' \) and \( R = R' \).

Let \( \Lambda = \mathbb{Z}_p[T_1, \ldots, T_r] \). We consider the ring homomorphisms \( \Lambda \to A \) defined by \( T_i \mapsto t_i \) and \( \Lambda \to \mathbb{Z}_p \) defined by \( T_i \mapsto 0 \). Then \( \mathbb{Z}_p \otimes_A \Lambda = R \) and \( \text{Tor}^1(\mathbb{Z}_p, A) = 0 \) for \( i > 0 \). Since \( R \) has no \( p \)-torsion it follows that \( \text{Tor}^1(\mathbb{F}_p, A) = 0 \) for \( i > 0 \).

Let \( \Lambda' = \mathbb{Z}_p(T_1, \ldots, T_r) \) be the divided power polynomial algebra and let \( \Lambda' = \mathbb{Z}_p \)-flat. For any \( x \in \Lambda' \) the divided power envelope of \( J \subseteq A \) with respect to \( \Sigma \). By the proof of [7, Pr. 3.4.4] we have \( \Lambda' = \mathbb{Z}_p \)-flat, and \( \Lambda'/p\Lambda' \) is isomorphic to a direct sum of copies of \( \mathbb{F}_p[T_1, \ldots, T_r]/(T_1^p, \ldots, T_r^p) \) as a \( \Lambda \)-module. Since the last module is a finite successive extension of copies of \( \mathbb{F}_p \), it follows that \( \text{Tor}^1(\Lambda'/p\Lambda', A) = 0 \). Therefore \( \Lambda' \) is \( \mathbb{Z}_p \)-flat, and thus \( S \) is \( \mathbb{Z}_p \)-flat.

From now on we assume that \( S \) is \( \mathbb{Z}_p \)-flat. Let \( \varphi : A \to A \) be a Frobenius lift. It extends naturally to an endomorphism \( \varphi : S \to S \) that induces the Frobenius on \( S/pS \) and makes \( S \) into a \( \mathbb{Z}_p \)-module. Let \( \text{Fil} S \) be the associated frame as in Remark 2.1.11, i.e. \( \varphi_1 = p^{-1}\varphi \) on \( \text{Fil} S \).

Assume that \( A_0 = A/pA \) has a finite \( p \)-adic basis \( (x_i) \). Then \( \widehat{\Omega}_A \) is a free \( A \)-module with basis \( (dx_i) \). The universal derivation \( d : A \to \widehat{\Omega}_A \) extends to a continuous connection

\[
(2.6.1) \quad \nabla_S : S \to S \otimes_A \widehat{\Omega}_A \quad \text{satisfying} \quad \nabla_S(x^{[j]}) = x^{[j-1]} \otimes dj
\]

for any \( x \in \text{Fil} S \); see [19, Remarks 2.2.4 (d)].

**Definition 2.6.2.** Let \( \text{Win}(S)^\nabla \) be the category of pairs \((M, \nabla)\) where \( M \) is an \( S \)-window and where \( \nabla : M \to M \otimes_A \widehat{\Omega}_A \) is a connection over \( \nabla_S \) for which \( \Phi \) is horizontal, i.e. \( \nabla \circ \Phi = (\Phi \otimes d\varphi) \circ \nabla \). Morphisms in \( \text{Win}(S)^\nabla \) are homomorphisms of \( S \)-windows that are compatible with the connections.

This definition is justified by the following lemma; cf. [29, Def. 3.2.3].

**Lemma 2.6.3.** For \((M, \nabla) \in \text{Win}(S)^\nabla \), the connection \( \nabla \) is integrable and topologically quasi-nilpotent.
Proposition 2.6.4. It follows that $D$ is divisible by $p$ for $x \in A$. The obvious homomorphism $\nabla^2 : M \to M \otimes_A \Omega_A$ satisfies $\nabla^2 \circ \varphi = (\varphi \otimes \Omega^2(\varphi)) \circ \nabla^2$. Since $p$ is not a zero divisor in $S$, it follows that $\nabla^2 \circ \varphi = p(\varphi \otimes \Omega^2(p^{-1}d\varphi)) \circ \nabla^2$ on $\Fil M$. Since the image of $\varphi$ generates $M$ as an $S$-module, we deduce that if $\nabla^2$ is divisible by $p^n$ for some $n \geq 0$, then $\nabla^2$ is divisible by $p^{n+1}$. Thus $\nabla^2 = 0$, i.e. $\nabla$ is integrable.

The proof that $\nabla$ is topologically quasi-nilpotent follows [19, 2.4.8]. Let $j : M \to M^{(\varphi)}$ be the $\varphi$-linear map $j(x) = 1 \otimes x$. There is a well-defined connection $\nabla^{(\varphi)} : M^{(\varphi)} \to M^{(\varphi)} \otimes_A \Omega_A$ over $\nabla S$ such that $\nabla^{(\varphi)}(j(x)) = (j \otimes d\varphi)(\nabla(x))$, and the homomorphisms $F : M^{(\varphi)} \to M$ and $V : M \to M^{(\varphi)}$ are horizontal with respect to $\nabla$ and $\nabla^{(\varphi)}$. Let $\vartheta : \Omega_A \to S$ be an $A$-linear map such that the associated derivation of $S/pS$ is nilpotent, and consider the associated differential operators $D = \vartheta \circ \nabla : M \to M$ and $D' = \vartheta \circ \nabla^{(\varphi)} : M^{(\varphi)} \to M^{(\varphi)}$. Since $d\varphi$ is divisible by $p$ we have $D'(1 \otimes x) \in pM^{(\varphi)}$. Thus $D'$ is nilpotent on $(M/pM)^{(\varphi)}$. Using the exact sequence with horizontal maps

$$
(M/pM)^{(\varphi)} \xrightarrow{F} M/pM \xrightarrow{V} (M/pM)^{(\varphi)}
$$

it follows that $D$ is nilpotent on $M/pM$, thus $\nabla$ is topologically quasi-nilpotent. \qed

Proposition 2.6.4. There is an exact equivalence of categories

$$
\DF(Spfr) \to \Win(S)^\nabla
$$

that is compatible with duality.

This is similar to [29, Pr. 3.2.5] (but there condition (2.4.3) is left out).

Proof. The functor in the proposition exists and is fully faithful by Proposition 2.5.2 together with the equivalence between crystals in finite locally free $\partial_{\Spf R/S}$-modules over $\Cris(Spfr/S)$ and finite projective $S$-modules with an integrable topologically quasi-nilpotent connection; see [19, Pr. 2.2.2]. It remains to verify that the functor is essentially surjective. Using Lemmas 2.5.1 and 2.6.3, this translates into the following assertion: If $(\mathcal{D}, F, V)$ is a Dieudonné crystal over $\Spf R$ and if $\Fil \mathcal{D}_R \subseteq \mathcal{D}_R$ is a locally direct summand such that condition (2.4.3) holds for $U = \Spec R/pR$ and $Z = \Spec S/pS$, then (2.4.3) holds for all $(U \to Z)$. We may assume that $U$ and $Z$ are affine. Since $A/pA$ has a $p$-basis, the given morphism $U \to \Spec R/pR \to \Spec A/pA$ extends to $Z$. Thus we obtain a morphism of PD-thickenings from $(U \to Z)$ to $(\Spec R/pR \to \Spec S/pS)$, and the assertion follows by pull back. \qed

Remark 2.6.5. The preceding proof shows that for the specific PD-frames considered here, the equivalent conditions of Lemma 2.5.1 imply the equality (2.4.3) for all $(U \to Z)$. This does not hold for arbitrary PD-frames.

2.7. Dieudonné theory over complete regular local rings. Let $R$ be a complete regular local ring with perfect residue field $k$ of characteristic $p \geq 3$ and with fraction field of characteristic zero. We put $\mathcal{S} = W(k)[[u_1, \ldots, u_d]]$ where $d = \dim(R)$ and choose a homomorphism $\pi : \mathcal{S} \to R$ such that the elements $\pi(u_i)$ generate the
maximal ideal of $R$. Then $R = \mathcal{S}/E\mathcal{S}$ for a power series $E \in \mathcal{S}$ with constant term of $p$-value one.

As in §2.3 let $S$ be the $p$-adic completion of the divided power envelope of the ideal $E\mathcal{S} \subset \mathcal{S}$. Let $\varphi : \tilde{\mathcal{S}} \to \mathcal{S}$ be a lift of Frobenius and denote its extension to $S$ by $\tilde{\varphi}$ again. The element $c = \varphi(E)/p$ is a unit of $S$; see [34, Le. 6.1]. We obtain a frame homomorphism $\lambda : \tilde{\mathcal{S}} \to S$, which is crystalline by Proposition 2.3.1.

Since $\varphi$ is a lift of Frobenius, its derivative $d\varphi : \tilde{\Omega}_\mathcal{S} \to \tilde{\Omega}_\mathcal{S}$ is divisible by $p$. Let $(d\varphi)_1 := p^{-1}d\varphi$ as an endomorphism of $\tilde{\Omega}_\mathcal{S}$.

**Proposition 2.7.1.** If $(d\varphi)_1$ is nilpotent on $\tilde{\Omega}_\mathcal{S} \otimes_\mathcal{S} k$, then the contravariant functor $\text{pdiv}(R) \to \text{Win}(S)$ given by the Dieudonné crystal (Propositions 2.4.3 and 2.5.2) is an equivalence of categories.

The hypothesis holds when $\varphi(u_i) = u_i^p$ because then $(d\varphi)_1$ is zero on $\tilde{\Omega}_\mathcal{S} \otimes_\mathcal{S} k$. In the case $d = 1$ with $\varphi(u_i) = u_i^p$, the proposition is due to Breuil [11, Th. 4.2.2.9]; alternative proofs are given by Kisin [30, Prop. A6] and by Zink [46, Prop. 3.7]. The general case does not follow from [46, Thm. 6] because the ring $S$ is a $\tilde{\mathcal{Z}}$-ring in the sense of [46] only when $d = 1$.

**Proof.** The proposition is a consequence of the results of [34], with a possible variation using Proposition 2.3.1. Let $\tilde{\mathcal{W}}(R) = \mathcal{W}(R)$ be the subring of $W(R)$ defined in [45], and let $\tilde{\mathcal{W}}(R) = (\mathcal{W}(R), 1_R, R, \sigma, \sigma_1)$ be the associated frame where $\sigma_1$ is the inverse of the Verschiebung; see [34, Sec. 2.3]. This is a PD-frame because $\mathcal{W}(R)$ is $p$-adically complete with no $p$-torsion by [34, Prop. 1.14], and the ideal $1_R$ carries divided powers by [34, Lemma 1.16].

We have natural frame homomorphisms

\[(2.7.1) \quad \mathcal{S} \xrightarrow{\lambda} S \xrightarrow{\tilde{\varphi}} \mathcal{W}(R)\]

where $\lambda$ is the inclusion. Namely, there is a natural ring homomorphism $\varkappa : \tilde{\mathcal{S}} \to W(R)$ compatible with the projections to $R$ such that $\sigma \varkappa = \varkappa \varphi$, and the image of $\varkappa$ lies in the subring $\tilde{\mathcal{W}}(R)$ if and only if $(d\varphi)_1$ is nilpotent on $\tilde{\Omega}_\mathcal{S} \otimes_\mathcal{S} k$; see [34, Prop. 6.2] (by a change of variables one can assume that $\varphi$ preserves the ideal of $\mathcal{S}$ generated by the $u_i$, which is assumed in loc.cit.). Since $\tilde{\mathcal{W}}(R)$ is a PD-frame, $\varkappa$ extends to a ring homomorphism $\tilde{\varkappa} : S \to \tilde{\mathcal{W}}(R)$, which is a strict frame homomorphism as in (2.7.1).

Now $\tilde{\varkappa}$ is crystalline by [34, Thm. 7.2]; alternatively one can use that $\lambda$ is crystalline by Proposition 2.3.1, and that $\varkappa = \tilde{\varkappa} \circ \lambda$ is crystalline by [34, Thm. 6.5]. Thus the proposition holds if and only if the composition

\[
\text{pdiv}(R) \longrightarrow \text{Win}(S) \xrightarrow{\tilde{\varkappa}^*} \text{Win}(\tilde{\mathcal{W}}(R))
\]

is an equivalence. This composition is given by evaluating the crystalline Dieudonné module as in Proposition 2.5.2, and the claim follows from the crystalline version of the equivalence between $p$-divisible groups and Dieudonné displays in [34, Cor. 5.4].

**Proposition 2.7.2.** If $(d\varphi)_1$ is nilpotent on $\tilde{\Omega}_\mathcal{S} \otimes_\mathcal{S} k$, then the forgetful functor $\text{Win}(S)^\nabla \to \text{Win}(S)$ is an equivalence of categories.
Proof. Let \( M \) be an \( \mathcal{S} \)-window. We have to show that there is a unique connection \( \nabla : M \to M \otimes_{\hat{\mathcal{O}}} \hat{\mathcal{O}} \) over \( \nabla_{\mathcal{S}} \) such that \( \nabla \circ \Phi = (\Phi \otimes d\varphi) \circ \nabla \). As earlier we define \( F : M^{(\varphi)} \to M \) and \( V : M \to M^{(\varphi)} \) by \( F(x \otimes 1) = \Phi(x) \) for \( x \in M \) and \( V(\Phi_1(y)) = y \otimes 1 \) for \( y \in \text{Fil} M \); see Remarks 2.1.4.

We choose a basis of \( M \) and denote by \( A \) and \( B \) the corresponding matrices of \( F \) and \( V \). We may assume that the basis of \( M \) lies in the image of \( \Phi_1 \). Then \( B = \varphi(\hat{B}) \) for another matrix \( \hat{B} \). Under the identification \( M \cong S^n \) we have \( \nabla(x) = \nabla_S(x) + C \cdot x \) for \( x \in S^n \), where \( C \) is a matrix with coefficients in \( S \otimes_{\mathcal{O}} \hat{\mathcal{O}} \). The condition on \( \nabla \) translates into the equation

\[
\nabla_S(A) + CA = A \cdot d\varphi(C)
\]

(here \( d\varphi \) is short for \( \varphi \otimes d\varphi \), or equivalently (after right multiplying by \( B \))

\[
\nabla_S(A) \cdot B + pC = pA \cdot (d\varphi)_1(C) \cdot B.
\]

Now \( AB = A\varphi(\hat{B}) = pE_n \) has zero derivative, thus

\[
\nabla_S(A) \cdot B = -pA \cdot (d\varphi)_1(\nabla_S(\hat{B})),
\]

and the condition becomes (after dividing by \( p \))

\[
C - \mathcal{U}(C) = D
\]

with \( \mathcal{U}(C) = A \cdot (d\varphi)_1(C) \cdot B \) and \( D = A \cdot (d\varphi)_1(\nabla_S(\hat{B})) \). Set \( y^{(n)} := \varphi^n(y) \). The \( n \)-th iterate of \( \mathcal{U} \) is given by

\[
C \mapsto AA^{(1)} \cdot A^{(n-1)}(d\varphi)_1^n(C)(B^{(n-1)} \cdots B^{(1)}B).
\]

Now we are facing the difficulty that the inclusion map \( \mathcal{S} \to S \) is continuous with respect to the \( \mathfrak{m}_\mathcal{S} \)-adic topology on \( \mathcal{S} \) and the \( p \)-adic topology on \( S \) only when \( d = 1 \). In that case, the hypothesis on \( (d\varphi)_1 \) implies that for large \( n \), the semilinear endomorphism \( (d\varphi)_1^n \) of \( S \otimes_{\mathcal{O}} \hat{\mathcal{O}} \) is divisible by \( p \). It follows that \( 1 - \mathcal{U} \) is bijective, so there is a unique solution \( C \).

For \( d \geq 2 \) we can argue as follows. Let \( T = \mathcal{S}[E^p/p] \subseteq S \). By Proposition 2.3.1 we may assume that \( M \) comes from an \( \mathcal{S} \)-window \( M' \). If the basis of \( M \) is chosen in \( p^{-1}\Phi(M') \), then \( A \) and \( B \) have coefficients in \( T \). Moreover, since \( \varphi(E) \in pT \) it is easy to see that \( \varphi : S \to S \) factors over \( \bar{\varphi} : S \to T \), and \( (d\varphi_1) \) induces a \( \bar{\varphi} \)-linear map \( S \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to T \otimes_{\mathcal{O}} \hat{\mathcal{O}} \). Thus the solutions of (2.7.2) over \( S \) and over \( T \) are the same. The hypothesis on \( (d\varphi)_1 \) implies that the semilinear endomorphism \( (d\varphi)_1^n \) of \( T \otimes_{\mathcal{O}} \hat{\mathcal{O}} \) is nilpotent modulo the maximal ideal of \( T \), so again there is a unique solution \( C \).

Corollary 2.7.3. The filtered crystalline Dieudonné module functor of Proposition 2.4.3 induces a contravariant equivalence of categories

\[
p\text{div}(R) \xrightarrow{\sim} \text{DF}(\text{Spf } R).
\]

Proof. Chose \( \varphi : \mathcal{S} \to \mathcal{S} \) such that \( (d\varphi)_1 \) is nilpotent on \( \hat{\mathcal{O}} \otimes k \), for example \( \varphi(u_i) = u_i^p \). We have a sequence of functors

\[
p\text{div}(R) \to \text{DF}(\text{Spf } R) \xrightarrow{\sim} \text{Win}(\mathcal{S})^\nabla \xrightarrow{\sim} \text{Win}(\mathcal{S})
\]
where the second and third arrow are equivalences by Propositions 2.6.4 and 2.7.2, and the total composition is an equivalence by Proposition 2.7.1. Thus the first arrow is an equivalence. 

\[ \square \]

3. DIEUDONNÉ THEORY OVER CYCLOTOMIC RINGS

In this section we will prove Theorem 1 in a slightly more general setting, allowing to replace the ring $O_K$ by a ring of power series over $O_K$. We begin by defining the setup and the relevant module categories.

3.1. Strict actions of the cyclotomic Galois group. Fix a perfect field $k$ of characteristic $p > 2$, and let $F = W(k) \otimes \mathbb{Q}$ and $F_s = F(\mu_{p^{s}})$ for $s \geq 0$ and $F_{\infty} = \bigcup_{s} F_s$. We choose a compatible system of primitive $p^{s}$-th roots of unity $\varepsilon^{(s)} \in F_s$. Fix $d \geq 0$.

For $s \geq 0$ we consider the surjective homomorphism of $W(k)$-algebras
\[ \mathfrak{S} := W(k)[u, t_1, \ldots, t_d] \xrightarrow{\pi_s} R_s := O_{F_s}[t_1, \ldots, t_d] \]
defined by $1 + u \mapsto \varepsilon^{(s)}$ and $t_i \mapsto t_i$. Let $\varphi : \mathfrak{S} \to \mathfrak{S}$ extend the Frobenius of $W(k)$ by $\varphi(1 + u) = (1 + u)^p$ and $\varphi(t_i) = t_i^p$. The kernel of $\pi_s$ is generated by the element $E_s$ defined by $E_0 = u$ and $E_s = \varphi^s(u)/\varphi^{s-1}(u)$ for $s \geq 1$.

Let $\Gamma_s = \text{Gal}(F_{\infty}/F_s)$ and let $\chi : \Gamma_0 \to \mathbb{Z}_p^\times$ be the $p$-adic cyclotomic character. We let $\Gamma_0$ act on the $W(k)$-algebra $\mathfrak{S}$ by $\gamma(1 + u) = (1 + u)^{\chi(\gamma)}$ and $\gamma(t_i) = t_i$. Then $\pi_s$ is equivariant with respect to the natural action of $\Gamma_0$ on $R_s$ that fixes all $t_i$. The action of $\Gamma_0$ on $\mathfrak{S}$ commutes with $\varphi$, and it is continuous in the following sense.

**Lemma 3.1.1.** Let $s \geq 0$. The action of $\Gamma_0$ on $\mathfrak{S}$ preserves the ideal $\varphi^s(u)\mathfrak{S}$, and the induced action of $\Gamma_s$ on $\mathfrak{S}/\varphi^s(u)\mathfrak{S}$ is trivial.

**Proof.** We have $\varphi^s(u) = E_0 E_1 \cdots E_s$ as a product of pairwise coprime prime elements of the factorial ring $\mathfrak{S}$, and for $0 \leq i \leq s$ the group $\Gamma_0$ acts on $R_i = \mathfrak{S}/E_i\mathfrak{S}$ with trivial action of $\Gamma_s$. The lemma follows easily. \[ \square \]

For $r \geq 1$ we consider the frame $\mathfrak{S}_r = (\mathfrak{S}_r, E_r, \mathfrak{S}_r, R_r, \varphi, \varphi_1)$ with $\mathfrak{S}_r = \mathfrak{S}$ and $\varphi'(E_r x) = \varphi(x)$. Here $\varphi_1$ depends on $r$. Each $\gamma \in \Gamma_0$ defines a window automorphism $\gamma : \mathfrak{S}_r \to \mathfrak{S}_r$, which is a $c_{r}$-automorphism with $c_r = \gamma(\varphi(E_r))/\varphi(E_r)$, i.e. we have $\varphi_1' \gamma = c_r \gamma \varphi_1'$. Let $S_r$ be the $p$-adic completion of the divided power envelope of the ideal $E_r\mathfrak{S}$ of $\mathfrak{S}$. We obtain a frame $\mathfrak{S}_r = (S_r, \text{Fil} S_r, R_r, \varphi, \varphi_1)$ with $\varphi_1 = p^{-1} \varphi$. The inclusion $\mathfrak{S}_r \to S_r$ is a $c$-homomorphism of frames with $c = \varphi(E_r)/p$. The action of $\Gamma_0$ on $\mathfrak{S}$ induces an action on $S_r$ by strict frame automorphisms.

**Remark 3.1.2.** It will be convenient to denote the variable $u$ of $\mathfrak{S}_r$ by $u_r$ and to consider $\mathfrak{S}_r$ as a subring of $\mathfrak{S}_{r+1}$ by letting $u_r = \varphi(u_{r+1})$. This makes $\mathfrak{S}_r$ into a subframe of $\mathfrak{S}_{r+1}$ extending the obvious inclusion $R_r \to R_{r+1}$. The elements $u_0 = \varphi^{r}(u_r)$ and $E = E_r(u_r) = u_0/u_1$ and $c_r$ are independent of $r$; moreover $c_r$ lies in $W(k)[u_0]$. Similarly, $S_r$ becomes a subframe of $S_{r+1}$, and $c$ is independent of $r$.

In the following let subgroups $\Gamma_r \subseteq \Gamma' \subseteq \Gamma \subseteq \Gamma_0$ be given with $r \geq 1$. We note that the projection $\mathfrak{S} \to \mathfrak{S}/u\mathfrak{S}$ maps $E_r$ into the divided power ideal $(p)$ and thus extends to a ring homomorphism $S_r \to \mathfrak{S}/u\mathfrak{S}$. 

Definition 3.1.3. An action of $\Gamma$ on an $\mathcal{S}_r$-window or $\mathcal{S}_\tau$-window $M$ by window automorphisms over the given action of $\Gamma$ on $\mathcal{S}_r$ or $\mathcal{S}_\tau$ is called $\Gamma'$-strict if the induced action of $\Gamma'$ on $M \otimes_\mathcal{S} \mathcal{S}/u\mathcal{S}$ or $M \otimes_{\mathcal{S}_r} \mathcal{S}/u\mathcal{S}$ is trivial. We denote by $\text{Win}(\mathcal{S}_r)^{\Gamma'}$ and $\text{Win}(\mathcal{S}_\tau)^{\Gamma'}$ the categories of windows with a $\Gamma'$-strict action of $\Gamma$.

Definition 3.1.5. An action of $\Gamma$ on a BT module $(\mathcal{M}, \varphi)$ over $\mathcal{S}_r$ by semilinear automorphisms is called $\Gamma'$-strict if the induced action of $\Gamma'$ on $\mathcal{M}/u\mathcal{M}$ is trivial. The category of BT modules over $\mathcal{S}_r$ with a $\Gamma'$-strict action of $\Gamma$ is denoted by $\text{BT}(\mathcal{S}_r)^{\Gamma'}$. Again we write $\text{BT}(\mathcal{S}_r)_{\Gamma} = \text{BT}(\mathcal{S}_r)^{\Gamma}_{\Gamma'}$ and $\text{BT}(\mathcal{S}_\tau)^{\Gamma} = \text{BT}(\mathcal{S}_\tau)^{\Gamma}_{\Gamma'}$.

Lemma 3.1.6. The equivalence of Lemma 2.1.15 extends to an exact equivalence

$$\text{Win}(\mathcal{S}_r)^{\Gamma'} \rightarrow \text{BT}(\mathcal{S}_r)^{\Gamma'}$$

where $\gamma \in \Gamma$ acts on $\mathcal{M} = \text{Fil} M$ by $E/\gamma(E) \cdot (\gamma_M^{\text{Fil}M})$.

Proof. Assume that $M \in \text{Win}(\mathcal{S}_r)$ and $\mathcal{M} \in \text{BT}(\mathcal{S}_r)$ correspond to each other, in particular $\mathcal{M} = \text{Fil} M$ and $M = \varphi^* \mathcal{M}$. If $(\mathcal{M}, \varphi)$ carries an action of $\Gamma$, we let $\gamma \in \Gamma$ act on $M$ by $\gamma_{\mathcal{M}} \otimes \gamma_{\mathcal{M}}$. One verifies that this definition and the construction of the lemma give well-defined and mutually inverse functors between $\text{Win}(\mathcal{S}_r)^{\Gamma'}$, and $\text{BT}(\mathcal{S}_r)^{\Gamma'}$.

Remark 3.1.7. In the definition of strict actions, we could pass from $\mathcal{S}/u\mathcal{S}$ to $W(k) = \mathcal{S}/(u, t_1, \ldots, t_d)$ without changing the resulting category; of course this is relevant only when $d \geq 1$. More precisely, in Definition 3.1.5 let $\mathcal{M}_0 = \mathcal{M}/u\mathcal{M}$ and let $\mathcal{M}_0 = \mathcal{M} \otimes_\mathcal{S} W(k)$. Then $\Gamma'$ acts trivially on $\mathcal{M}_0$ if and only it acts trivially on $\mathcal{M}_0$.

This follows from the fact that the reduction map $\text{End}(\mathcal{M}_0, \varphi) \rightarrow \text{End}(\mathcal{M}_0, \varphi)$ is injective. Indeed, let $J = (t_1, \ldots, t_d)\mathcal{S}/u\mathcal{S}$, choose a basis of $\mathcal{M}_0$, and let $A$ be the corresponding matrix over $\mathcal{S}/u\mathcal{S}$ of $\varphi$. Then $pA^{-1}$ exists in $\mathcal{S}/u\mathcal{S}$. An endomorphism of $(\mathcal{M}_0, \varphi)$ is given by a matrix $C$ that satisfies $CA = A\varphi(C)$, or equivalently $pC = A\varphi(C)pA^{-1}$. Assume that $C$ has coefficients in $J$. Since $\varphi(J) \subseteq J^p$ and since each $(\mathcal{S}/u\mathcal{S})/J^m$ is torsion free, it follows that the coefficients of $C$ lie in $J^m$ for all $m \geq 1$, and thus $C = 0$. 

Proposition 3.1.4. The base change functor $\text{Win}(\mathcal{S}_r)^{\Gamma'} \rightarrow \text{Win}(\mathcal{S}_r)^{\Gamma'}$ is an exact equivalence of categories.

Proof. This is a consequence of Proposition 2.3.1, using that an action of $\Gamma$ on an $\mathcal{S}_r$-window or $\mathcal{S}_\tau$-window $M$ can be given by window isomorphisms $\gamma^*(M) \rightarrow M$ for $\gamma \in \Gamma$; see (2.1.1). The strictness condition is preserved when passing from $\mathcal{S}_r$ to $\mathcal{S}_\tau$. □
Similarly, in Definition 3.1.3 let \( M_0 = M \otimes S / u S \) or \( M_0 = M \otimes S_r S / u S \), and let \( \overline{M}_0 = M_0 \otimes S / u S W(k) \). Then \( \Gamma' \) acts trivially on \( M_0 \) if and only if it acts trivially on \( \overline{M}_0 \); this follows by the proofs of Lemma 3.1.6 and Proposition 3.1.4.

Remark 3.1.8. The duality of windows and BT modules extends naturally to a duality of such objects equipped with a \( \Gamma' \)-strict action of \( \Gamma \). For \( M \in \text{Win}(S)_{\Gamma'} \) the associated action on \( M' \) is simply the contragredient action defined by \( \gamma(f)(\gamma(m)) = f(m) \) for \( m \in M \) and \( f \in M' \). Over \( S_r \), a twist occurs: For \( \gamma \in \Gamma_0 \), the infinite product

\[
\lambda_\gamma = \prod_{n \geq 0} \varphi^n(E/\gamma(E))
\]

converges in \( S_r \) and is independent of \( r \). For \( M \in \text{BT}(S)_{\Gamma'} \), we define an action on \( M' \) by \( \gamma(f)(\gamma(m)) = \lambda_\gamma \cdot \gamma(f(m)) \), while for \( M \in \text{Win}(S)_{\Gamma'} \) we define an action on \( M' \) by \( \gamma(f)(\gamma(m)) = \varphi(\lambda_\gamma) \cdot \gamma(f(m)) \). One can verify that these definitions give \( \Gamma' \)-strict actions of \( \Gamma \) and that the equivalences of Lemma 3.1.6 and Proposition 3.1.4 preserve duality.

When \( d \geq 1 \) we also need the following variant:

Definition 3.1.9. Let \( \Omega_0 := \Omega_{S/W(k)}[t_1, \ldots, t_d] = S du \), and let \( \nabla_{S_0,0} : S_r \to S_r \otimes S \Omega_0 \) be the composition of \( \nabla_S \) and the natural map \( \Omega_S \to \Omega_0 \). We denote by \( \text{Win}(S)_{\nabla_0}^{\nabla} \) the category of \( S_r \)-windows \( M \) equipped with a connection \( \nabla_0 : M \to M \otimes S \Omega_0 \) over \( \nabla_{S_0,0} \) with respect to which \( \Phi \) is horizontal. Let \( \text{Win}(S)_{\Gamma_0}^{\nabla, \Gamma} \) be the category of objects of \( \text{Win}(S)_{\nabla_0}^{\nabla} \) with a \( \Gamma' \)-strict and horizontal action of \( \Gamma \).

Lemma 3.1.10. The functors

\[
\text{Win}(S)_{\nabla}^{\nabla} \to \text{Win}(S)_{\nabla_0}^{\nabla_0} \quad \text{and} \quad \text{Win}(S)_{\nabla_0}^{\nabla, \Gamma} \to \text{Win}(S)_{\Gamma_0}^{\nabla, \Gamma}
\]

defined by composing a connection \( M \to M \otimes \tilde{\Omega}_S \) with the natural map \( \tilde{\Omega}_S \to \Omega_0 \) are equivalences.

Proof. We have \( \tilde{\Omega}_S = \Omega_0 \oplus \Omega_1 \) with \( \Omega_1 = \tilde{\Omega}_S / W(k)[u] \). The endomorphism \( (d\varphi)_1 = p^{-1}d\varphi \) of \( \tilde{\Omega}_S \) preserves this decomposition, and its restriction to \( \Omega_1 \) is topologically nilpotent due to the choice \( \varphi(t_i) = t_i^p \). The lemma now follows from the proof of Proposition 2.7.2.

3.2. Construction of strict actions. Next we study how \( p \)-divisible groups, or equivalently filtered \( S_r \)-stabilized \( D \)\textsuperscript{-}crystals over \( R_r \) are related to windows with strict actions of \( \Gamma_r \). Recall that \( r \geq 1 \).

Lemma 3.2.1. The evaluation functor of Proposition 2.6.4 extends to an exact functor

\[
\text{DF}(\text{Spf } R_r) \to \text{Win}(S)_{\nabla, \Gamma_r}
\]

that preserves duality.
Proof. Let \( \mathcal{D} \in \text{DF}(\text{Spf}(R_r)) \) map to \((M, \nabla) \in \text{Win}(\mathcal{S}_r)^{\nabla, \Gamma_r}\), so in particular \( M = \mathcal{D}_{/u_r} \). Since each \( \gamma \in \Gamma_r \) defines an endomorphism of the PD-extension \( S_r \to R_r \) that commutes with \( \varphi \) and is trivial on \( R_r \), we obtain a horizontal action of \( \Gamma_r \) on \( M \). The action is strict because \( M \otimes S_r \mathcal{G} / u \mathcal{G} \) is the value of \( \mathcal{D} \) at the PD-extension \( \mathcal{G} / u \mathcal{G} = W(k)[t_1, \ldots, t_d] \to k[t_1, \ldots, t_d] \), which is a quotient of \( S_r \to R_r \) on which \( \Gamma_r \) acts trivially. \( \square \)

Proposition 3.2.2. The natural functors

\[
\text{pdiv}(R_r) \to \text{DF}(\text{Spf}(R_r)) \to \text{Win}(\mathcal{S}_r)^{\nabla, \Gamma_r} \xrightarrow{j} \text{Win}(\mathcal{S}_r)^{\nabla, 0}
\]

are all equivalences of categories. (The first functor is contravariant.)

Proof. The first arrow is an equivalence by Corollary 2.7.3. The second arrow exists by Lemma 3.2.1, and its composition with the forgetful functor \( j \) is an equivalence by Proposition 2.6.4 and Lemma 3.1.10. It remains to show that \( j \) is fully faithful. More precisely we show that for an object \((M, \nabla_0) \) of \( \text{Win}(\mathcal{S}_r)^{\nabla, 0}\) and for \( \gamma \in \Gamma_r \) there is at most one horizontal automorphism \( \gamma_M : M \to M \) over the automorphism \( \gamma \) of \( S_r \), such that \( \gamma_M \) induces the identity on \( M \otimes S_r \mathcal{G} / u \mathcal{G} \), or equivalently on \( M \otimes S_r W(k) \); see Remark 3.1.7. If the given pair \((M, \nabla_0) \) corresponds to \( G \in \text{pdiv}(R_r) \), by the composite equivalence, then \( M \otimes S_r W(k) \) corresponds to the special fibre \( G_k \) by classical Dieudonné theory, and \( \gamma_M \) corresponds to an automorphism of \( G \) that induces the identity of \( G_k \). Thus \( \gamma_M \) is unique by the rigidity of \( p \)-divisible groups; see [39, II.3.3.21] and its proof. \( \square \)

Lemma 3.2.3. The forgetful functor \( \text{Win}(\mathcal{S}_r)^{\nabla, \Gamma_r} \to \text{Win}(\mathcal{S}_r)^{\nabla, 0} \) is fully faithful.

Proof. We have to show that for each \( \underline{M} \in \text{Win}(\mathcal{S}_r)^{\nabla, \Gamma_r} \) there is at most one \( \nabla_0 : M \to M \otimes \Omega_0 \) which makes \( \underline{M} \) into an object of \( \text{Win}(\mathcal{S}_r)^{\nabla, \Gamma_r} \). The evaluation of \( \nabla_0 \) at \((u + 1)(d/du)\) gives a differential operator \( N : M \to M \) with \( N \circ \gamma = \chi(\gamma) \gamma \circ N \) for \( \gamma \in \Gamma_r \). The difference of two choices of \( \nabla_0 \) gives an \( S \)-linear map \( \Delta : M \to M \) with \( \Delta \circ \gamma = \chi(\gamma) \gamma \circ \Delta \). Let \( L = \text{Frac}(W(k)[t_1, \ldots, t_d]) \) and \( S' = L[u] \). Then \( \Gamma_r \) acts on \( S' \), and we have an injective equivariant homomorphism \( S_r \to S' \). Let \( \Delta' : M' \to M' \) be the scalar extension of \( \Delta \) to \( S' \). If \( \Delta \neq 0 \) we choose \( n \geq 0 \) maximal such that \( \Delta'(M') \subseteq u^n M' \). Then \( \Delta' \) induces a non-zero \( L \)-linear map \( \overline{\Delta} : M'/uM' \to u^n M'/u^{n+1} M' \). Since the group \( \Gamma_r \) acts strictly on \( M \), it acts trivially on \( M'/uM' \) and acts via \( \chi^n \) on \( u^n M'/u^{n+1} M' \). Since for \( \gamma \neq 1 \) the equation \( \chi^{n+1}(\gamma) a = a \) has no non-zero solution \( a \in L \), it follows that \( \overline{\Delta} = 0 \), a contradiction. Thus \( \Delta = 0 \), and \( \nabla_0 \) is unique. \( \square \)

3.3. Construction of the connection. We want to show that the forgetful functor of Lemma 3.2.3 is essentially surjective, i.e. that every \( \mathcal{S}_r \)-window with a strict \( \Gamma_r \)-action carries a natural connection. To this end, we need some preparations. As earlier let \( u_0 = \varphi^\ast(u_r) \), which is independent of \( r \). We fix \( r \geq 1 \) and write \( u := u_r \) and \( \mathcal{G} := \mathcal{G}_r \) for simplicity.

Lemma 3.3.1. For \( \underline{M} \in \text{Win}(\mathcal{S}_r)^{\nabla, \Gamma_r} \) the induced action of \( \Gamma_r \) on \( M/u_0 M \) is trivial.
Proof. Let \( \gamma \in \Gamma_r \). We show by induction that for \( 0 \leq s \leq r \) we have \( (\gamma - 1)M \subseteq \varphi^s(u)M \). The case \( s = r \) is the assertion of the lemma, while the case \( s = 0 \) is the definition of a strict action. Let \( s \geq 1 \) and assume that \( (\gamma - 1)M \subseteq \varphi^{s-1}(u)M \).

\[
N = \{ x \in M \mid \gamma(x) - x \in \varphi^s(u)M \} = \{ x \in M \mid \gamma(x) - x \in E_sM \};
\]

here the second equality holds because \( \varphi^s(u) = \varphi^{s-1}(u) \cdot E_s \) is a product of coprime factors in the factorial ring \( \mathcal{G} \). For \( x \in M \) and \( a \in \mathcal{G} \) we have

\[
(3.3.1) \quad (\gamma - 1)(ax) = (\gamma - 1)(a) \cdot x + \gamma(a) \cdot (\gamma - 1)(x).
\]

Using Lemma 3.1.1 it follows that \( N \) is an \( \mathcal{G} \)-module. As \( \gamma \) commutes with \( \Phi \), the inductive hypothesis implies that \( \Phi(M) \subseteq N \). Since \( \varphi(E_r)M \subseteq \mathcal{G} \cdot \Phi(M) \) it follows that \( \varphi(E_r)M \subseteq N \). Using (3.3.1) with \( a = \varphi(E_r) \), for each \( x \in M \) we get that \( \gamma(a) \cdot (\gamma - 1)(x) \in E_sM \). Since \( a \notin E_s\mathcal{G} \) and thus \( \gamma(a) \notin E_s\mathcal{G} \), it follows that \( (\gamma - 1)(x) \in E_sM \) as required.

We consider the element \( t := \log(1 + u_0) \in S_r \), which exists since \( u_0 = \varphi^{r-1}(u)E_r \) lies in \( \text{Fil} S_r \), and which is independent of \( r \). A direct calculation shows that in \( S_r \) we have

\[
t = \lim_{n \to \infty} \frac{\varphi^n(u_0)}{p^n} = u_0 \prod_{n \geq 1} \frac{\varphi^n(E_r)}{p} = u_0 \cdot \text{unit}.
\]

We have \( \varphi(t) = pt \) and \( \gamma(t) = \chi(\gamma)t \) for \( \gamma \in \Gamma_0 \). It is easy to see that \( u_0^{p-1} \in pS_r \) and thus \( t^{p-1} \in pS_r \).

Lemma 3.3.2. For \( \gamma \in \Gamma_r \) we have \( (\gamma - 1)(S_r) \subseteq tS_r \).

Proof. We have \( (\gamma - 1)(\mathcal{G}) \subseteq u_0\mathcal{G} \subseteq tS_r \) by Lemma 3.1.1. Assume that for some \( a \in \text{Fil} S_r \) we have \( \gamma(a) - a = bt \) with \( b \in S_r \). Then \( \gamma(a^p) = (a + bt)^p = a^p + pxt + b^p t^p \) for some \( x \in S_r \). Since \( t^{p-1} \in pS_r \) it follows that \( (\gamma - 1)(a^p/p) \in tS_r \). Since the divided power envelope of the ideal \( E_r \mathcal{G} \subseteq \mathcal{G} \) is generated as an \( \mathcal{G} \)-algebra by the successive iterates of \( E_r \) under \( a \mapsto a^p/p \), the lemma follows.

Lemma 3.3.3. Let \( M \in \text{Win}(S_r)^{\Gamma_r} \).

1. For \( \gamma \in \Gamma_r \) and \( n \geq 0 \) we have \( (\gamma - 1)^{n+1}(M) \in (t, p^n)^n tM \).
2. For \( \gamma \in \Gamma_{r+n} \) with \( n \geq 0 \) we have \( (\gamma - 1)(M) \subseteq (t, p^n)^n tM \).
3. For each \( n \geq 0 \) the action of \( \Gamma_r \) on \( M/p^n M \) has an open kernel.

In particular, the action of \( \Gamma_r \) on \( M \) is continuous.

Proof. By Proposition 3.1.4, \( M \) comes from an object \( M' \in \text{Win}(\mathcal{G}_r)^{\Gamma_r} \), in particular \( M = S_r \otimes_{\mathcal{G}_r} M' \). Using (3.3.1) with \( a \in S_r \) and \( x \in M' \), Lemmas 3.3.1 and 3.3.2 give the case \( n = 0 \) of both (1) and (2). A simple induction, using again (3.3.1) and the relation \( \gamma(t) = \chi(\gamma)t \), gives (1) in general. Since the multiplication \( p : \Gamma_{r+n} \to \Gamma_{r+n+1} \) is bijective and

\[
(\gamma^p - 1) = (p + \sum_{i=1}^{p-1} (\gamma^i - 1))(\gamma - 1),
\]

a similar induction gives (2) in general. Assertion (3) follows from (2) since \( t^{p-1} \in pS_r \). \( \square \)
Following [2, §4.1] one can differentiate a continuous action of $\Gamma_r$ on a finite free $S_r$-module $M$ as follows. For $\gamma \in \Gamma_r$ sufficiently close to 1 we have $(\gamma - 1)(M) \subseteq pM$, which implies that for such $\gamma$ the series
\[
\log(\gamma)(x) = \sum_{n \geq 1} (-1)^{n-1} \frac{(\gamma - 1)^n(x)}{n}
\]
with $x \in M$ converges. For $a \in \mathbb{Z}_p$ we have $\log(\gamma^a) = a \cdot \log(\gamma)$, and thus
\[
N_M := p^r \log(\gamma) / \log(\chi(\gamma))
\]
is a well-defined map $N_M : M \to M \otimes \mathbb{Q}$ which is independent of $\gamma$ sufficiently close to 1. It is easy to see that
\[
N_M = p^r \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1}.
\]
It follows that $N_M(ax) = N_S(a)x + a N_M(x)$ for $a \in S_r$ and $x \in M$, in particular $N_S$ is a derivation, and $N_M$ is a differential operator over $N_S$. Since $N_S(t) = p't$ we must have $N_S = (1+u)t(d/du)$. For $\gamma \in \Gamma_r$ we have $N_M \gamma = \gamma N_M$.

In our context the main point is that for strict actions on windows, no denominators occur:

**Proposition 3.3.4.** For $M \in \text{Win}(S_r)^{\Gamma_r}$ we have $N_M(M) \subseteq tM$ and $N_M \Phi = \Phi N_M$.

**Proof.** Since $t^{p-1} \in p S_r$, Lemma 3.3.3 (1) implies that the series $\log(\gamma)$ converges for every $\gamma \in \Gamma_r$ and that $\log(\gamma)(M) \subseteq tM$. Since the factor $p^r / \log(\chi(\gamma))$ is a $p$-adic unit when $\gamma \in \Gamma_r \setminus \Gamma_{r+1}$, the first assertion follows. The second assertion is clear. $\square$

**Corollary 3.3.5.** The forgetful functor $\text{Win}(S_r)^{\nabla_0^{\Gamma_r}} \to \text{Win}(S_r)^{\Gamma_r}$ is an equivalence of categories.

**Proof.** The functor is fully faithful by Lemma 3.2.3. Given $M \in \text{Win}(S_r)^{\Gamma_r}$, let $\nabla_0 : M \to M \otimes_{\Omega} \Omega_0$ be the connection whose evaluation at $(1+u)t(d/du)$ is the differential operator $N_M$ of Lemma 3.3.4; this is well-defined since the image of $N_M$ lies in $tM$. Since the derivation $(1+u)t(d/du) : S_r \to S_r$ commutes with $\Phi$ and with all $\gamma \in \Gamma_r$, the relations $N_M \Phi = \Phi N_M$ and $N_M \gamma = \gamma N_M$ mean that $\Phi$ and the action of $\Gamma_r$ are horizontal with respect to $\nabla_0$. Thus $(M, \nabla_0)$ is an object of $\text{Win}(S_r)^{\nabla_0^{\Gamma_r}}$, which proves that the functor is essentially surjective. $\square$

Together with Proposition 3.2.2 we obtain:

**Corollary 3.3.6.** There is a contravariant and exact equivalence of categories $\text{pdiv}(R_r) \cong \text{Win}(S_r)^{\Gamma_r}$ that preserves duality. $\square$
3.4. $p$-divisible groups with descent. The preceding results can be extended quite formally to include the following objects. Let $F \subseteq K \subseteq K'$ be finite extensions. We put $R = \mathcal{O}_K[t_1, \ldots, t_d]$ and $R' = \mathcal{O}_{K'}[t_1, \ldots, t_d]$.

**Definition 3.4.1.** Let $\text{pdiv}(R')^K$ be the category of $p$-divisible groups $G$ over $R'$ equipped with a descent of $G \otimes_{R'} R'[p^{-1}]$ to $R[p^{-1}]$. If $K'$ is contained in $F_\infty$ we also write $\text{pdiv}(R')^K = \text{pdiv}(R')^{\Gamma_K}$.

**Remark 3.4.2.** Assume that $K'/K$ is Galois with Galois group $\Gamma$, for example $K' \subseteq F_\infty$ and $\Gamma = \Gamma_K/\Gamma_{K'}$. Then $\text{pdiv}(R')^K$ is equivalent to the category of $G \in \text{pdiv}(R')$ that are equipped with isomorphisms $G_{R'[p^{-1}]} \cong \tau^*(G_{R'[p^{-1}]})$ for each $\tau \in \overline{\Gamma}$ satisfying the obvious cocycle condition. By Tate’s theorem, these isomorphisms extend uniquely to isomorphisms $G \cong \tau^*(G)$ over $R'$.

**Proposition 3.4.3.** Assume that $F \subseteq K \subseteq F_r$ with $r \geq 1$, and let $\Gamma = \Gamma_K$. There are exact equivalences of categories (the first one contravariant)

$$\text{pdiv}(R_r)^\Gamma \cong \text{Win}(S_r)^\Gamma \cong \text{Win}(\mathfrak{S}_r)^\Gamma \cong \text{BT}(\mathfrak{S}_r)^\Gamma$$

that preserve duality.

**Proof.** The second and third equivalence are Proposition 3.1.4 and Lemma 3.1.6. To prove the first equivalence, let $G \in \text{pdiv}(R_r)$, and let $M(G) \in \text{Win}(S_r)^\Gamma_r$ be the associated $\Gamma_r$-window given by Corollary 3.3.6. Let us put $\overline{\Gamma} = \Gamma/\Gamma_r$. By Remark 3.4.2, giving a descent of $G \otimes R_r[p^{-1}]$ to $R[p^{-1}]$ is equivalent to giving isomorphisms $w_\tau : G \to \tau^*(G)$ for each $\tau \in \overline{\Gamma}$ that satisfy the cocycle condition. Every $\gamma \in \Gamma$ that lifts $\tau$ acts on the PD-frame $S_r$, and we have a natural isomorphism $\gamma^*M(G) \cong M(\tau^*G)$. Thus $w_\tau$ induces an isomorphism $w_\gamma : \gamma^*(M(G)) \to M(G)$. In this way, we obtain a bijection between descent data for $G \otimes R_r[p^{-1}]$ relative to $\overline{\Gamma}$ and actions of $\Gamma$ on $M(G)$ that extend the given action of $\Gamma_r$, and the first equivalence of the proposition follows from that of Corollary 3.3.6.

To extend this result further, we consider a chain $F \subseteq K \subseteq K' \subseteq K'' \subseteq F_r$ of extensions with $r \geq 1$. Let $\Gamma'' \subseteq \Gamma' \subseteq \Gamma$ be the corresponding groups, and let $R'' = \mathcal{O}_{K''}[t_1, \ldots, t_d]$. We note that for $G \in \text{pdiv}(R'')^\Gamma$, the isomorphisms of Remark 3.4.2 give an action of $\Gamma'/\Gamma''$ on the special fibre $G_k$.

**Lemma 3.4.4.** The base change functor $\text{pdiv}(R')^\Gamma \to \text{pdiv}(R'')^\Gamma$ is fully faithful, and its essential image consists of all $G \in \text{pdiv}(R'')^\Gamma$ for which the action of $\Gamma'/\Gamma''$ on $G_k$ is trivial.

**Proof.** One easily reduces to the case $K = K'$, which leaves us with the functor $\text{pdiv}(R) \to \text{pdiv}(R'')^\Gamma$. Clearly the functor is fully faithful, and for $G$ in its image the action of $\Gamma'/\Gamma''$ on $G_k$ is trivial. For a given $p$-divisible group $G_k$ over $k$ let $A$ be its universal deformation ring over $W(k)$. A deformation of $G_k$ to $G \in \text{pdiv}(R'')$ corresponds to a homomorphism of $W(k)$-algebras $h : A \to R''$, and there are isomorphisms $G \cong \tau^*(G)$ which reduce to the identity of $G_k$ for each $\tau \in \Gamma'/\Gamma''$ if and only if for each $\tau$ we have $\tau \circ h = h$. This means that the image of $h$ lies in the ring of invariants $(R'')^{\Gamma'/\Gamma''} = R$, i.e. $G$ is defined over $R$. □
Again let $F \subseteq K \subseteq K' \subseteq F_r$ with $r \geq 1$ and let $\Gamma = \Gamma_K$ and $\Gamma' = \Gamma_{K'}$.

**Proposition 3.4.5.** There are exact equivalences of categories

$$\text{pd}_{\text{div}}(R')^\Gamma \cong \text{Win}((\mathcal{S}_r)^{\Gamma}, \mathcal{S}_r, \lambda) \cong \text{Win}((\mathcal{S}_{\tilde{r}})^{\Gamma}, \mathcal{S}_{\tilde{r}}, \lambda) \cong \text{BT}(\mathcal{S}_r)^{\Gamma},$$

(the first one contravariant) that preserve duality.

**Proof.** This is a consequence of Proposition 3.4.3 together with Lemma 3.4.4 applied with $K'' = F_r$. Namely, let $G \in \text{pd}_{\text{div}}(R)^\Gamma$ correspond to the $\Gamma$-window $M$ over $\mathcal{S}_r$. By Remark 3.1.7, the action of $\Gamma$ is trivial on $M/uM$ if and only if it is trivial on $M \otimes_\mathbb{F} W(k)$, which means that $\Gamma'/\Gamma$ acts trivially on $G_k$, i.e. that $G$ lies in the image of $\text{pd}_{\text{div}}(R')^\Gamma$. This gives the first equivalence. The second and third equivalence are straightforward. $\square$

For $K = K'$ we obtain the following version of Theorem 1:

**Corollary 3.4.6.** The category $\text{pd}_{\text{div}}(R)$ is equivalent to the category $\text{BT}(\mathcal{S}_r)^\Gamma$ of BT modules $\mathcal{M}$ over $\mathcal{S}_r$, with an action of $\Gamma_K$ which is trivial on $\mathcal{M}/u\mathcal{M}$.

**Remark 3.4.7.** Assume that $F \subseteq \tilde{K} \subseteq K' \subseteq F_{\tilde{r}}$ is another instance of the above data with $K \subseteq \tilde{K}$ and $K' \subseteq \tilde{K}'$ and $r \leq \tilde{r}$. Under the equivalence of Proposition 3.4.5, the base change functor $\text{pd}_{\text{div}}(R')^\Gamma \rightarrow \text{pd}_{\text{div}}(\tilde{R})^{\tilde{\Gamma}}$ corresponds to the base change functor $\text{BT}(\mathcal{S}_r)^{\Gamma} \rightarrow \text{BT}(\mathcal{S}_{\tilde{r}})^{\tilde{\Gamma}}$, defined by $(\mathcal{M}, \varphi) \mapsto (\mathcal{M} \otimes_\mathcal{S}_r \mathcal{S}_{\tilde{r}}, \varphi \otimes \varphi)$ with the obvious action of $\tilde{\Gamma}$. Here $\mathcal{S}_r$ is viewed as a subring of $\mathcal{S}_{\tilde{r}}$ as in Remark 3.1.2.

**Example 3.4.8.** Let us trace the constructions for the $p$-divisible groups $\mathbb{Q}_p/\mathbb{Z}_p$ and $\mathbb{G}_m$. The $\mathcal{S}_r$-window associated to $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mathcal{S}_r = (S_r, \text{Fil} S_r, R_r, \varphi, \varphi_1)$ with the standard action of $\Gamma_0$, which corresponds to the $\mathcal{S}_{\tilde{r}}$-window $\mathcal{S}_{\tilde{r}} = (\mathcal{S}_{\tilde{r}}, E_{\tilde{r}} \mathcal{S}_{\tilde{r}}, \varphi, \varphi')$ and the BT module $(\mathcal{S}_{\tilde{r}}, \varphi)$, both with the standard action of $\Gamma_0$. The $\mathcal{S}_r$-window associated to $\mathbb{G}_m$ is $\mathcal{S}_L^r = (S_r, S_r, p, \varphi, \varphi)$, and we claim that this corresponds to the $\mathcal{S}_{\tilde{r}}$-window $\mathcal{S}_{\tilde{r}} = (\mathcal{S}_{\tilde{r}}, \varphi(E_{\tilde{r}}) \varphi, \varphi)$ with $\gamma \in \Gamma_0$ acting as $x \mapsto \varphi(\lambda_\gamma x)$, where $\lambda_\gamma$ is defined in (3.1.1). Indeed, the base change of $\mathcal{S}_r^L$ to $\mathcal{S}_{\tilde{r}}$ is equal to $(\mathcal{S}_{\tilde{r}})^L_e = (S_r, S_r, \varphi(E_r) \varphi, \varphi)$, and multiplication by $u_0/t$ carries this window isomorphically onto $\mathcal{S}_r^L$. The associated BT module is $(\mathcal{S}_r, E_r \varphi)$ with $\gamma \in \Gamma_0$ acting as $x \mapsto \lambda_\gamma x$.

4. **Galois representations**

In this section we relate Theorem 1 with the theory of Galois representations. We return to the setting of Theorem 1, which corresponds to the case $d = 0$ of §3.

Recall that $F = W(k) \otimes \mathbb{Q}$ for a perfect field $k$ of odd characteristic $p$. For a finite extension $K$ of $F$ let $\mathcal{G}_K$ be its absolute Galois group, let $\mathcal{H}_K \subset \mathcal{G}_K$ be the kernel of the $p$-adic cyclotomic character, and $\Gamma_K = \mathcal{G}_K/\mathcal{H}_K$. We will always assume that $K \subseteq F_r = F(\mu_{p^r})$ for some $r$, which means that $\mathcal{H}_K = \mathcal{H}_{F_r}$. The main point will be to recover the Tate module of a $p$-divisible group $G$ over $\mathcal{O}_K$ as a representation of $\mathcal{G}_K$ from the BT module with $\Gamma_K$-action associated to $G$ by the crystalline theory. This can be done by a variant of Faltings’ [22, Th. 7].
4.1. **Rings.** To fix the notation we recall some of Fontaine’s rings. Let $C_F$ be the completion of the algebraic closure of $F$, and let $R := \lim_{\to} (O_{C_F}/p, \varphi)$ and $A_{\inf} := W(R)$, with $\theta : A_{\inf} \to O_{C_F}$ the unique homomorphism that lifts the natural map $A_{\inf} \to O_{C_F}/p$. Let $A_{\text{cris}}$ be the $p$-adic completion of the PD-envelope of $R$. These rings carry a Frobenius $\varphi$ and an action of $G_F$. The chosen system of primitive $p^r$-th roots of unity $\varepsilon^{(r)} \in O_{C_F}$ corresponds to an element $\xi$ of $R$. We put $u_0 = [\xi] - 1 \in A_{\inf}$ and $u_r = \varphi^{-r}(u_0)$ and consider $\mathcal{S}_r = W(k)[u_r]$ as a subring of $A_{\inf}$. Here $\varphi$ and an element $g \in G_F$ act by $\varphi(1 + u_r) = (1 + u_r)^p$ and $g(1 + u_r) = (1 + u_r)^{\chi(g)}$. Let $O_{\mathcal{S}_r}$ be the $p$-adic completion of $\mathcal{S}_r[1/u]$, viewed as a subring of $W(Fr(R))$, let $O_{\mathcal{S}_r}^{\text{sep}} \subseteq W(Fr(R))$ be the completion of its maximal unramified extension, and $\mathcal{S}_r^{\text{nr}} = O_{\mathcal{S}_r}^{\text{nr}} \cap A_{\inf}$. We have $\mathcal{S}_F = \text{Gal}(\mathcal{S}_r/\mathcal{S}_r^{\text{nr}})$.

Sometimes we drop the index $r = 0$ and write $u = u_0$, $\mathcal{S} = \mathcal{S}_0$, etc.

The restriction of $\theta$ to $\mathcal{S}_r$ is the homomorphism $\theta_r : \mathcal{S}_r \to O_{F_r}$ defined by $1 + u_r \mapsto \varepsilon^{(r)}$. For $r \geq 1$, the element $E = E_r(u_r) = u_0/\varphi^{-1}(u_0)$ is independent of $r$, and it generates the ideals $\text{Ker}(\theta)$ and $\text{Ker}(\theta_r)$. Let $\theta_r^{nr} : \mathcal{S}_r^{nr} \to O_{C_F}$ be the restriction of $\theta_r$.

**Lemma 4.1.1.** For $r \geq 1$, $\theta_r^{nr}$ induces an isomorphism $\mathcal{S}_r^{nr}/E \cong O_{C_F}$.

*Proof.* Since $\text{Ker}(\theta)$ is generated by $E$ and since the multiplication by $E$ is bijective on $W(Fr(R))$ and on $O_{\mathcal{S}_r}$, we have $\text{Ker}(\theta_r^{nr}) = E\mathcal{S}_r^{nr}$. Let $O_{\mathcal{S}_r} = k[u_r]$ and let $O_{\mathcal{S}_r}^{\text{sep}}$ be the integral closure of $O_{\mathcal{S}_r}$ in the separable closure of $k((u_r))$ in Fr(R). The inclusion $\mathcal{S}_r^{nr} \to W(R)$ taken mod $p$ gives the inclusion $O_{\mathcal{S}_r}^{\text{sep}} \to R$, which becomes bijective after $u_r$-adic completion by [23, A.3.1.6]. Therefore the image of $\mathcal{S}_r^{nr} \to O_{C_F}/p$, which is $O_{\mathcal{S}_r}^{\text{sep}}/u_r^{p-1}$, coincides with $R/u_r^{p-1} = O_{C_F}/p$. Since $\mathcal{S}_r^{nr}$ is $p$-adically complete it follows that $\theta_r^{nr}$ is surjective. \hfill $\Box$

For $r \geq 1$ let $S_r$ and $S_r^{nr}$ be the $p$-adic completions of the PD-envelopes of $E\mathcal{S}_r$ and $E\mathcal{S}_r^{nr}$. The construction of §2.3 applied to $\mathcal{S}_r$, $\mathcal{S}_r^{nr}$, and $A_{\inf}$ gives a commutative diagram of frames

\[(4.1.1)\quad \xymatrix{ \mathcal{S}_r & \mathcal{S}_r^{nr} & A_{\inf} \\ \mathcal{S}_r^{nr} & A_{\text{cris}} }\]

where in the upper row, the ideals Fil are generated by $E$ with $\varphi'_1(Ex) = \varphi(x)$, and in the lower row, the ideals Fil are the natural PD-ideals with $\varphi_1 = p^{-1}\varphi$. The horizontal arrows are strict homomorphisms, and the vertical arrows are $c$-homomorphisms for $c = \varphi(E)/p$ which induce equivalences on windows by the descent Proposition 2.3.1. The Galois group $G_F$ acts compatibly on all frames of the diagram, more precisely $g \in G_F$ induces strict automorphisms of the lower three frames and $c_g$-automorphisms of the upper three frames with $c_g = g(\varphi(E))/\varphi(E)$. The action of $G_F$ on $\mathcal{S}_r$ and on $\mathcal{S}_r^{nr}$ factors over $\Gamma_F$. Here the nested system $(\mathcal{S}_r \to \mathcal{S}_r)_r \geq 1$ coincides with that considered in §3 for $d = 0$; see Remark 3.1.2.
4.2. Recovering the Tate module. Let $F \subseteq K \subseteq K' \subseteq F_r$ with $r \geq 1$ be given. For $G \in \operatorname{pdiv}(\mathcal{O}_{K'})^{\Gamma_K}$ the Tate module $T_p(G)$ is naturally a $\mathbb{Z}_p[G_K]$-module, which we want to reconstruct from the BT module with $\Gamma_K$-action $\mathbb{M}(G)$ associated to $G$ by Proposition 3.4.5. Since the base change functor $\operatorname{pdiv}(\mathcal{O}_{K'})^{\Gamma_K} \to \operatorname{pdiv}(\mathcal{O}_{F_r})^{\Gamma_K}$ changes neither $T_p(G)$ nor $\mathbb{M}(G)$, for simplicity we may and do assume that $K' = F_r$. In particular, $\operatorname{pdiv}(\mathcal{O}_{K'})$ is viewed as a full subcategory of $\operatorname{pdiv}(\mathcal{O}_{F_r})^{\Gamma_K}$.

In the following we will use the convention that a BT module $\mathbb{M} \in \operatorname{BT}(\mathcal{G}_r)$ corresponds to the windows $\mathcal{M} \in \operatorname{Win}(\mathcal{G}_r)$ and $\mathcal{M} \in \operatorname{Win}(\mathcal{S}_r)$, moreover $\mathcal{M}^{nr}$, $\mathcal{M}^{\inf}$, $\mathcal{M}^{nr}$, $\mathcal{M}^{\inf}$, and $\mathcal{M}^{\text{cris}}$ denote the obvious base change to the other four frames of (4.1.1); see Lemma 2.1.15 and Proposition 2.3.1.

For $\mathcal{M} \in \operatorname{BT}(\mathcal{G}_r)^{\Gamma_K}$ as in Definition 3.1.5 we consider the $\mathbb{Z}_p[G_K]$-module

$$T^r_p(\mathcal{M}) = \operatorname{Hom}_{\mathcal{G}_r,\varphi}(\mathcal{M}, \mathcal{S}_r^{nr})$$

where $g \in G_K$ acts by conjugation. Note that $T^r_p(\mathcal{M}) = \operatorname{Hom}(\mathcal{M}^{nr}, \mathcal{S}_r^{nr})$ in the category $\operatorname{BT}(\mathcal{S}_r^{nr})$.

**Lemma 4.2.1.** There are natural $G_K$-equivariant isomorphisms

$$T^r_p(\mathcal{M}) \cong \operatorname{Hom}_{\mathcal{G}_r,\varphi}(\mathcal{M}, A^{nr}) \cong \operatorname{Hom}(\mathcal{M}^{nr}, \mathcal{S}_r^{nr}) \cong \operatorname{Hom}(\mathcal{M}^{\text{cris}}, A^{\text{cris}}),$$

and $T^r_p(\mathcal{M})$ is a free $\mathbb{Z}_p$-module of rank equal to the rank of $\mathcal{M}$ over $\mathcal{S}_r$.

**Proof.** We have $\operatorname{Hom}_{\mathcal{G}_r,\varphi}(\mathcal{M}, \mathcal{O}_{\mathcal{S}_r^{nr}}) = \operatorname{Hom}_{\mathcal{S}_r,\varphi}(\mathcal{M}, W(\operatorname{Fr}(R)))$, and this is a free $\mathbb{Z}_p$-module of rank equal to the rank of $\mathcal{M}$; see [23, A.1.2.7 and A.2.1.3]. Since $W(\operatorname{Fr}(R))/A^{nr}$ has no non-zero $\varphi$-stable and $\mathcal{S}_r$-finite submodules (cf. [23, B.1.8.4]), this Hom module coincides with $\operatorname{Hom}_{\mathcal{G}_r,\varphi}(\mathcal{M}, A^{nr})$ and therefore also with $T^r_p(\mathcal{M})$.

By Lemma 2.1.15 and Example 2.1.16 applied to $\mathcal{S}_r^{nr}$ and $A^{nr}$ it follows that $T^r_p(\mathcal{M}) = \operatorname{Hom}(\mathcal{M}^{nr}, \mathcal{S}_r^{nr}) = \operatorname{Hom}(\mathcal{M}^{\inf}, A^{nr})$. The homomorphisms over $\mathcal{S}_r^{nr}$ and $A^{nr}$ are the same by Proposition 2.3.1. □

For $G \in \operatorname{pdiv}(R_r)^{\Gamma_K}$, the module and windows associated to $G$ in Proposition 3.4.3 will be denoted by $\mathbb{M}(G) \in \operatorname{BT}(\mathcal{M}_r)^{\Gamma_K}$ and $M(G) \in \operatorname{Win}(\mathcal{M}_r)^{\Gamma_K}$ and $M(G) \in \operatorname{Win}(\mathcal{S}_r)^{\Gamma_K}$. The action of $\Gamma_K$ induces an action of $G_K$ on the various base change modules and windows over the frames of (4.1.1). We consider the homomorphism of $\mathbb{Z}_p[G_K]$-modules

$$\alpha_G : T_p(G) \to T^r_p(\mathbb{M}(G))$$

defined as the composition

$$T_p(G) = \operatorname{Hom}_{\mathcal{O}_{p^e}}(\mathcal{Q}_p/\mathcal{Z}_p, G) \xrightarrow{M^{\text{cris}}} \operatorname{Hom}(\mathcal{M}^{\text{cris}}(G), \mathcal{M}^{\text{cris}}(\mathcal{Q}_p/\mathcal{Z}_p)) \cong T^r_p(\mathbb{M}(G))$$

using Example 3.4.8 and Lemma 4.2.1. Let

$$\tilde{\alpha}_G : T_p(G) \otimes_{\mathbb{Z}_p} \mathcal{S}_r^{nr} \to \operatorname{Hom}_{\mathcal{S}_r^{nr}}(\mathcal{M}^{nr}(G), \mathcal{S}_r^{nr})$$
be the $\mathcal{S}_r^{nr}$-linear map induced by $\alpha_G$, which is also $\varphi$ and $G_K$-equivariant.

**Proposition 4.2.2.** Here $\tilde{\alpha}_G$ is injective with cokernel annihilated by $u_1$, and $\alpha_G$ is bijective.
Proof. This is a variant of [22, Thm. 7]. As in loc.cit., the first assertion implies the second, but the second assertion is also a direct consequence of the isomorphism $T_p(G) \cong \text{Hom}(M_{\text{cris}}(G), A_{\text{cris}})$ proved in loc.cit. together with Lemma 4.2.1.

For the first assertion we start with the case $G = \hat{G}_m$, with associated windows as given in Example 3.4.8. The $\mathbb{Z}_p$-module $T_p(\hat{G}_m)$ is generated by $\varepsilon : Q_p/\mathbb{Z}_p \rightarrow \hat{G}_m$, which induces $t : (A_{\text{cris}})^t \rightarrow A_{\text{cris}}$ over $A_{\text{cris}}$. The corresponding homomorphism of $\mathbb{G}_{\text{nr}}$-windows $(\mathbb{G}_{\text{nr}})^t \rightarrow \mathbb{G}_{\text{nr}}$ is multiplication by $t \cdot u_0/t = u_0$, and the associated map of BT modules $M_{\text{nr}}^1(\hat{G}_m) \rightarrow M_{\text{nr}}^1(Q_p/\mathbb{Z}_p)$ is $\varphi^{-1}(u_0) = u_1 : \mathbb{G}_{\text{nr}} \rightarrow \mathbb{G}_{\text{nr}}$.

The general case follows by a standard duality argument. Let us denote by $\rho_G$ the “dual version” of $\alpha_G$, defined as the composite

$$T_p(G^\vee) = \text{Hom}_{\mathcal{O}_F}(G, \mathbb{G}_m) \rightarrow \text{Hom}(M_{\text{cris}}(\hat{G}_m), M_{\text{cris}}(G)) \cong \text{Hom}((\mathbb{G}_{\text{nr}})^t; M_{\text{nr}}(G))$$

where the last isomorphism follows from Lemma 4.2.1 applied to the dual windows. Let

$$\tilde{\rho}_G : T_p(G^\vee) \otimes \mathbb{Z}_p \mathbb{G}_{\text{nr}} \rightarrow \text{Hom}_{\mathbb{G}_{\text{nr}}}(\mathbb{G}_{\text{nr}}, M_{\text{nr}}(G))$$

be the associated $\mathbb{G}_{\text{nr}}$-linear map. The commutative diagram with perfect bilinear vertical maps

$$T_p(G) \times T_p(G^\vee) \xrightarrow{\alpha \times \rho} \text{Hom}_{\mathbb{G}_{\text{nr}}}(M_{\text{nr}}, \mathbb{G}_{\text{nr}}) \times \text{Hom}_{\mathbb{G}_{\text{nr}}}(\mathbb{G}_{\text{nr}}, M_{\text{nr}})$$

implies that $(\tilde{\rho}_G)^t \circ \alpha_G$ is $u_1$ times an invertible map. Since $T_pG$ and $M_{\text{nr}}(G)$ have equal rank, the proposition follows. We note that by the crystalline duality theorem, $\rho_G$ can be identified with $\alpha_{G^\vee}$, but this is not used here. \qed

4.3. Modules of finite $E$-height. Let $K$ be a finite extension of $F$ contained in $F_{\omega}$ and let $r \geq 0$. We denote by $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_K)$ the category of free $\mathbb{Z}_p$-representations of $\mathcal{G}_K$ and by $\text{Mod}_{\mathcal{O}_{\mathcal{E}_r}}(\varphi, \Gamma_K)$ the category of free etale $(\varphi, \Gamma_K)$-modules over $\mathcal{O}_{\mathcal{E}_r}$.

By Fontaine [23], we have mutually inverse equivalences of categories

$$\text{Rep}_{\mathbb{Z}_p}(G_F) \cong \text{Mod}_{\mathcal{O}_{\mathcal{E}_r}}(\varphi, \Gamma_K)$$

defined by $D_r(T) = (T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathcal{E}_r}})_{\delta_F}$ and $T_r(M) = (M \otimes_{\mathcal{O}_{\mathcal{E}_r}} \mathcal{O}_{\hat{\mathcal{E}_r}})^{\psi = 1}$.

Remark 4.3.1. For $0 \leq s \leq r$ these functors are related as follows. Let $i : \mathcal{O}_{\mathcal{E}_s} \rightarrow \mathcal{O}_{\mathcal{E}_r}$ be the inclusion. The scalar extension functor

$$(4.3.1) \quad i^* : \text{Mod}_{\mathcal{O}_{\mathcal{E}_s}}(\varphi, \Gamma_K) \rightarrow \text{Mod}_{\mathcal{O}_{\mathcal{E}_r}}(\varphi, \Gamma_K)$$

Let us recall the proof. The universal vector extension of $Q_p/\mathbb{Z}_p$ is obtained as the pushout of $\mathbb{Z}_p \rightarrow Q_p \rightarrow Q_p/\mathbb{Z}_p$ by $\mathbb{Z}_p \rightarrow \mathbb{G}_m$. The given map $\varepsilon : Q_p/\mathbb{Z}_p \rightarrow \mathbb{G}_m$ over $\mathcal{O}_{\mathcal{E}_r}$ lifts uniquely to a map $Q_p \rightarrow \mathbb{G}_m$ over $A_{\text{cris}}$ with $1 \mapsto [\varepsilon]$. Its restriction to $\mathbb{Z}_p$ extends to an algebraic homomorphism $\mathbb{G}_a \rightarrow \mathbb{G}_m$ over $\text{Spf} A_{\text{cris}}$, and $\text{Lie} \mathbb{G}_a \rightarrow \text{Lie} \mathbb{G}_m$ sends $1 = \log_{\mathbb{G}_a}(1)$ to $t = \log_{\mathbb{G}_m}(1)$. 
Proof. For given $M \otimes_{O} (\cdot)$ is fully faithful. Moreover, the Frobenius iterate $\varphi^{r-s}$ of $O_{E_{r}}$ induces a bijective homomorphism

$$\lambda_{r,s} : O_{E_{r}} \rightarrow O_{E_{s}}$$

with $i \circ \lambda_{r,s} = \varphi^{r-s}$, which induces an isomorphism $\lambda_{r,s}^{*} \circ D_{r} \cong D_{s}$.

**Definition 4.3.2.** For $r \geq 1$ (so $E \in \mathfrak{S}_{r}$) we write $\text{Mod}_{r} (\varphi, \Gamma_{E_{r}})$ for the category of finite free $\mathfrak{S}_{r}$-modules $\mathfrak{M}$ equipped with a $\varphi$-linear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}[E^{-1}]$ that induces an isomorphism $\mathfrak{M}[E^{-1}] \cong \mathfrak{M}[E^{-1}]$, and with an action of $\Gamma_{E_{r}}$ which commutes with $\varphi_{\mathfrak{M}}$ and which is finite on $\mathfrak{M}/u_{r}\mathfrak{M}$, i.e. an open subgroup acts trivially on this quotient.

Assume that $F \subseteq K \subseteq K' \subseteq F_{r}$ with $r \geq 1$. The main result of Kisin-Ren [31], specialised to the case where the Lubin-Tate group is the multiplicative group $G_{m}$, gives an equivalence $T \sim \mathfrak{M}(T)$ between the category of all $T \in \text{Rep}_{Z_{p}}(G_{K'})$ which are crystalline over $G_{K'}$ and the category of all $\mathfrak{M} \in \text{Mod}_{r} (\varphi, \Gamma_{K})$ where $\Gamma_{K'}$ acts trivially on $\mathfrak{M}/u_{r}\mathfrak{M}$, and one has a natural isomorphism $\mathfrak{M}(T) \otimes_{O_{E_{r}}} O_{E_{s}} \cong D_{r}(T)$.

Strictly speaking, this is stated in [31] only when $K = K'$, but the general case is a formal consequence. Moreover, [31] works with $\mathfrak{S} = \mathfrak{S}_{0}$ instead of $\mathfrak{S}_{r}$ and constructs a functor $T \sim \mathfrak{M}_{\text{KR}}(T)$ valued in the category $\text{Mod}_{r} (\varphi, \Gamma_{K})$ of $(\varphi, \Gamma_{K})$-modules over $\mathfrak{S}$ defined just as in Definition 4.3.2, replacing $\mathfrak{S}_{r}$ with $\mathfrak{S}$ and $E = E_{r}(u_{r})$ with $E_{r}(u) = E_{r}(u_{0})$. This makes no difference to the theory, as $\varphi^{r}$ induces a bijective homomorphism $\lambda_{r,0} : O_{E_{r}} \rightarrow O_{E}$ carrying $\mathfrak{S}_{r}$ onto $\mathfrak{S}$ and sending $E$ to $E_{r}(u)$, so by base change induces an equivalence of categories $\text{Mod}_{\mathfrak{S}_{r}} (\varphi, \Gamma_{K}) \sim \text{Mod}_{r} (\varphi, \Gamma_{K})$, and we define $\mathfrak{M}(T)$ as the inverse image of $\mathfrak{M}_{\text{KR}}(T)$ under this equivalence, in other words $\mathfrak{M}(T) := \mathfrak{S}_{r} \otimes_{\lambda_{r,0}^{-1} O_{E_{r}}} \mathfrak{M}_{\text{KR}}(T)$.

Moreover, by [30, Th. 0.3] the category of such $T$ with Hodge-Tate weights 0 and 1 is equivalent to the category $\text{pdv}(O_{K})^{1 \Gamma_{K}}$. Thus Theorem 1 is a special case of [31]. Let us verify that in this case the modules of [31] and of Theorem 1 are indeed the same up to a duality.

**Lemma 4.3.3.** The scalar extension functor

$$\text{Mod}_{\mathfrak{S}_{r}} (\varphi, \Gamma_{K}) \rightarrow \text{Mod}_{O_{E_{r}}} (\varphi, \Gamma_{K})$$

is fully faithful.

We note that this does not hold without a finite action of $\Gamma_{K}$. For example, $u_{1} : (\mathfrak{S}_{r}, E\varphi) \rightarrow (\mathfrak{S}_{r}, \varphi)$ becomes an isomorphism over $O_{E_{r}}$ but is not an isomorphism.

**Proof.** For given $\mathfrak{M}$ and $\widetilde{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}_{r}} (\varphi, \Gamma_{K})$ we write $M := \mathfrak{M} \otimes O_{E_{r}}$ and $\widetilde{M} := \widetilde{\mathfrak{M}} \otimes O_{E_{r}}$. The assertion is that every $(\varphi, \Gamma_{K})$-module homomorphism $f : \widetilde{M} \rightarrow M$ maps $\mathfrak{M}$ into $\mathfrak{M}$. Since for $0 \leq r \leq s$ we have $\mathfrak{S}_{r} = \mathfrak{S}_{s} \cap O_{E_{r}}$ and since (4.3.1) is an equivalence, we may increase $r$ and thus assume that $K \subseteq F_{r}$ and that $\Gamma_{F_{r}}$ acts trivially on $\mathfrak{M}/u_{r}\mathfrak{M}$ and on $\mathfrak{M}'/u_{r}\mathfrak{M}'$. In that case the assertion follows from [31, Cor. 3.3.8].

**Remark 4.3.4.** Lemma 4.3.3 is analogous to [30, Prop. 2.1.12] and [14, Prop. 3.1], and one can easily give a direct proof along similar lines. Namely, for $f : \widetilde{M} \rightarrow M$
as above, using [30, 2.1.10] it follows that \( \mathcal{W}' = f(\mathcal{M}) + \mathcal{M} \) is torsion free with \( \mathcal{W} \otimes \mathcal{O}_E, = M \). One can replace \( \mathcal{M} \) by \( \mathcal{W}'[1/p] \cap M \) and thus assume that \( M = \mathcal{M} \) with \( f = \text{id} \) and \( \mathcal{W} \subseteq \mathcal{M} \). In order to show that \( \mathcal{W} = \mathcal{M} \) one can pass to the determinant. Since \( \mathcal{M} \) and \( \mathcal{M} \) have finite \( E \)-height, we find \( \mathcal{M} = u_n^0 \mathcal{M} \) for some \( n \geq 0 \), using for example [4, Le. 2.1.2]. The action of \( \Gamma_K \) can be finite on \( \mathcal{M}/u\mathcal{M} \) and on \( \mathcal{M}/n^0 \mathcal{M} \) only when \( n = 0 \).

For a given \( G \in \text{pdiv}(\mathcal{O}_K)^{\Gamma_K} \), let \( \mathcal{M}(G) \in \text{BT}(\mathcal{G}_p)^{\Gamma_K} \) be the module associated to it by Proposition 3.4.5. The dual of the homomorphism \( \alpha_G \) of Proposition 4.2.2 induces a \( \varphi \) and \( \mathcal{G}_K \)-equivariant isomorphism

\[
\mathcal{M}(G) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^s} \cong T_p(G)^{\vee} \otimes_{\mathcal{O}_E} \mathcal{O}_{\mathcal{E}^s}.
\]

The invariants under \( \mathcal{H}_K = \mathcal{H}_F \) give an isomorphism of \( (\varphi, \Gamma_K) \)-modules

\[
\mathcal{M}(G) \otimes_{\mathcal{O}_E} \mathcal{O}_{E^s} \cong D_r(T_p(G)^{\vee}).
\]

Thus by Lemma 4.3.3 we have

\[
\mathcal{M}(G) = \mathcal{M}(T_p(G)^{\vee})
\]
as submodules of \( D_r(T_p(G)^{\vee}) \) by the given embeddings.

4.4. Representations of finite height and Wach modules. For \( r \geq 1 \) and \( F \subseteq K \subseteq K' \subseteq F_r \), the Wach modules of Berger [3] and Berger–Breuil [4] provide a variant of the description of Kisin–Ren [31] of stable \( \mathcal{O}_p \)-lattices in \( \mathcal{G}_K \)-representations whose restriction to \( \mathcal{G}_K' \) is crystalline. As this variant figures prominently in applications (e.g. [1], [12], [16], [20], [21], [35], [36], [37]), for the sake of completeness we now recall the relation between [31] and [3, 4].

For \( T \in \text{Rep}_{\mathcal{G}_p}(\mathcal{G}_K) \) let \( D^+(T) = (T \otimes_{\mathcal{G}_p} \mathcal{G}^{\text{ur}})^{\mathcal{G}_{p}^r} \). It is well-known that this is a free \( (\varphi, \Gamma_K) \)-module over \( \mathcal{G} \) of rank \( \leq \) the rank of \( T \); see Lemma 4.4.1 below applied to \( M = D(T) \) and \( \mathcal{G}^{\text{ur}} = T \otimes_{\mathcal{G}_p} \mathcal{G}^{\text{ur}} \). In fact, \( D^+(T) \) is the maximal \( \varphi \)-stable finitely generated \( \mathcal{G} \)-submodule of \( D(T) \), which is denoted \( f_*(D(T)) \) in [23, B.1.4]; this holds since \( \mathcal{O}_{\mathcal{G}_{p}^r}/\mathcal{G}^{\text{ur}} \) has no non-zero \( \varphi \)-stable finitely generated \( \mathcal{G}^{\text{ur}} \)-submodule.

The representation \( T \) is called of finite height if the rank of \( D^+(T) \) is equal to the rank of \( T \), which implies that \( D^+(T) \otimes_{\mathcal{G}} \mathcal{O}_{E} \cong D(T) \). By a slight abuse of terminology, we call \( T \) crystalline if its restriction to \( \mathcal{G}_{F_r} \) is crystalline for some \( r \). If \( T \) is of finite height and de Rham, then \( T \) is crystalline by [42, §A.5]. Conversely, crystalline implies finite height by [4, Th. 2.5.3].

Assume that \( T \) is of finite height and de Rham with non-positive Hodge-Tate weights. Let \( V = T[p^{-1}] \). Then [42, §A.5] gives a free \( \mathcal{G} \)-submodule \( N \subseteq D^+(V) \) of rank equal to the rank of \( V \) and stable under \( \Gamma_K \) such that the action of \( \Gamma_K \) is finite on \( N/uN \). By [4, Th. 3.1.1] there is a unique choice \( N(V) \) of \( N \) such that \( D^+(V)/N(V) \) is annihilated by a power of \( u \). This is the maximal choice of \( N \), and \( N(V) \) is stable under \( \varphi \). Explicitly, starting with any \( N \) we have

\[
N(V) = D^+(V) \cap N[\epsilon(u)^{-1}]_{n \geq 1}.
\]
Moreover $N(T) = N(V) \cap D^+(T)$ is a free $\mathcal{G}$-module in $D^+(T)$ with analogous properties; see [3, Le. II.1.3]. The modules $N(V)$ and $N(T)$ are called the Wach modules of $V$ and of $T$, respectively.

Second, assume that $K \subseteq F_r$ and that $T$ becomes crystalline over $\mathcal{G}_F$, for fixed $r \geq 1$, again with non-positive Hodge-Tate weights. Then [31] implies again that $T$ is of finite height and that there is a unique choice $N_{KR}(V)$ of $N$ which is stable under $\phi$ and such that $N_{KR}(V)/\phi^*N_{KR}(V)$ is annihilated by a power of $E_r(u)$. More precisely, $N_{KR}(T) = N_{KR}(V) \cap D^+(T)$ is the module provided by [31, Cor. 3.3.8].

One can go back and forth between these modules as follows. If $N_{KR}(V)$ is given, then $N(V)$ is given by (4.4.1) with $N = N_{KR}(V)$. In fact, using [4, Le. 2.1.2] one can see that $D^+(V)/N_{KR}(V)$ is annihilated by a power of $E_1(u) \cdots E_r(u)$, so it suffices to take $1 \leq n \leq r$ in (4.4.1). Conversely, if $N(V)$ is given, then the quotient $N(V)/\phi^*N(V)$ is annihilated by a power of $E_1(u) \cdots E_r(u)$ by [4, Cor. 3.2.6]. Using this, one can see that starting with $N = N(V)$, after $r - 1$ iterations of the operation $N \mapsto \phi^*(N)[E_r(u)^{-1}] \cap N$ one arrives at $M_{KR}(V)$. In particular, when $r = 1$ we have $N(V) = N_{KR}(V)$.

**Lemma 4.4.1.** Let $M$ be a finite free $\mathcal{O}_E$-module and $\mathfrak{M}^{ur}$ a finite free $\mathcal{G}^{ur}$-module, both of rank $d$, together with an isomorphism $M \otimes_{\mathcal{O}_E} \mathcal{G}^{ur} \cong \mathfrak{M}^{ur} \otimes \mathcal{G}^{ur} \mathcal{G}^{ur}$. Then the $\mathcal{G}$-module $\mathfrak{M} = M \cap \mathfrak{M}^{ur}$ is free of rank $\leq d$. If the rank is equal to $d$ then $\mathfrak{M} \otimes_{\mathcal{G}} \mathcal{O}_E = M$.

**Proof.** Cf. [17, Lemme III.3]. Let us first consider an analogous question modulo $p$. Let $\overline{M} = M/p$, a vector space over $E = \mathcal{O}_E/p$, and $\overline{\mathfrak{M}} = \mathfrak{M}^{ur}/p$, a free $\mathcal{O}^{ur}$-module, and put $\overline{\mathfrak{M}} = \overline{M} \cap \overline{\mathfrak{M}}^{ur}$, an $\mathcal{O}_E$-module. Then $\overline{\mathfrak{M}}$ contains a basis of $\overline{M}$, and we claim that $\overline{\mathfrak{M}}$ is free of rank $d$. If not, then $\overline{\mathfrak{M}}$ is not finitely generated, and we find a strictly ascending sequence $N_1 \subset N_2 \subset \cdots \subset N$ of free submodules $N_i$ of rank $d$. By passing to the determinant we see that this cannot exist, which proves the claim. Now the image of $\mathfrak{M} \rightarrow M/p^a$ is isomorphic to $\mathfrak{M}_a = \mathfrak{M}/p^a$, and we have $\mathfrak{M} = \varprojlim_n \mathfrak{M}_n$. Since $\mathfrak{M}_1 \subseteq \overline{\mathfrak{M}}$, the module $\mathfrak{M}_1$ is free of rank $\leq d$. A basis of $\mathfrak{M}_1$ lifts to a basis of $\mathfrak{M}$. The final assertion is clear. \[\Box\]

**References**


