ABSTRACT. Using the theory of residues and Fourier Analysis, we derive many examples of modular forms, including the Dedekind Eta function. After proving several results of Ramanujan, we introduce a function that generalizes $\operatorname{sech}(\sqrt{\frac{1}{2}\pi z})$ in that it is its own Fourier Transform and it reduces to the above as a special case. We show how this leads to a proof of the functional equation $L(s,\chi)$, and then generalize several results of Ramanujan on self-reciprocal functions. Finally, we show how contour integration may be used to derive explicit product representations for j functions of particular subgroups of $GL_2(Z)$.

On the Transformation of Infinite Series

BRYDEN R. CAIS

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1. RESIDUE SUMMATION

Fix $t \in \Re$ and consider the function

$$f(z) = \frac{\pi \cot(\pi z)\coth(\pi z t)}{z^{2m+1}}.$$

$f(z)$ has simple poles at $z = n$ and $z = \frac{ni}{t}$ for each nonzero integer n, and a pole of order $2m+3$ at $z = 0$. Let $R_a(f)$ denote the residue of $f(x)$ at $x = a$. A simple calculation gives

$$\begin{aligned} R_n(f) &= \frac{\coth(\pi n t)}{n^{2m+1}}, \\ R_{\frac{ni}{t}}(f) &= (-1)^{m+1} t^{2m} \frac{\coth(\frac{\pi n}{t})}{n^{2m+1}}. \end{aligned}$$

Recalling the familar Laurent expansion of $\pi\cot(\pi z)$, we compute

$$R_0(f) = 2(2\pi)^{2m+1} \sum_{n=0}^{m+1} \frac{(-1)^n B_{2n} B_{2(m+1-n)}}{(2n)!(2(m+1-n))!} t^{2(m-n)+1}.$$

Hence, by the theory of residues we have

$$\sum_{n=1}^{\infty} \left(\frac{\coth(\pi n t)}{n^{2m+1}} + (-1)^{m+1} t^{2m} \frac{\coth(\frac{\pi n}{t})}{n^{2m+1}} \right) = -(2\pi)^{2m+1} \sum_{n=0}^{m+1} \frac{(-1)^n B_{2n} B_{2(m+1-n)}}{(2n)!(2(m+1-n))!} t^{2(m-n)+1} \tag{1}$$

[3,p155] Similarly, if $\varphi(z)$ is any meromorphic function such that $\varphi(z) = O(|z|^{-k})$ as $|z| \to \infty$ where $k > 1$, we can find transformation formulae for

$$\sum_{n=-\infty}^{\infty} \varphi(n)\coth(\pi n).$$

Theorem 1. *If*

$$f(\alpha, \beta) = \frac{1}{16\pi^2 u^4} + \sum_{n=1}^{\infty} \frac{n\coth(\alpha n)}{\alpha n^4 + 4\beta u^4},$$

then if $\alpha\beta = \pi^2$,

$$\begin{aligned} f(\alpha,\beta) \quad + \quad f(\beta,\alpha) = \frac{1}{4u^2} \{ & \frac{\cosh^2 u(\sqrt{\pi\alpha} + \sqrt{\pi\beta}) - \cos^2 u(\sqrt{\pi\alpha} + \sqrt{\pi\beta})}{(\cosh(2u\sqrt{\pi\alpha}) - \cos(2u\sqrt{\pi\alpha}))(\cosh(2u\sqrt{\pi\beta}) - \cos(2u\sqrt{\pi\beta}))} \\ & - \frac{\cosh^2 u(\sqrt{\pi\alpha} - \sqrt{\pi\beta}) - \cos^2 u(\sqrt{\pi\alpha} - \sqrt{\pi\beta})}{(\cosh(2u\sqrt{\pi\alpha}) - \cos(2u\sqrt{\pi\alpha}))(\cosh(2u\sqrt{\pi\beta}) - \cos(2u\sqrt{\pi\beta}))} \}. \end{aligned} \tag{2}$$

Proof. Consider the function $\psi(z) = \frac{\pi z \cot(\pi z)\coth(\alpha z)}{\alpha z^4 + 4\beta u^4}$. $\psi(z)$ has only simple poles. The residues at $z = n$ and $z = \frac{\pi n i}{\alpha}$ are easily calculated for each integer n, while the residues at $z = \zeta\sqrt{2}(\frac{\beta}{\alpha})^{\frac{1}{4}}$ for ζ a primitive eighth root of unity are computed using the elamentary result

$$\cot(x + iy) = \frac{\sin(2x) - i\sinh(2y)}{\cosh(2y) - \cos(2x)}. \tag{3}$$

This is due to Ramanujan. [4,p277] ■

Theorem 2. *If*

$$f(t,x) = \frac{1}{4tx^4} + \sum_{n=1}^{\infty} \frac{n\coth(\pi n t)}{n^4 + n^2x^2 + x^4},$$

then

$$f(t,x) + t^2 f(\frac{1}{t}, itx) = \frac{\pi}{x^2\sqrt{3}} \frac{\sinh(\pi x\sqrt{3})\sinh(\pi tx) + \sin(\pi tx\sqrt{3})\sin(\pi x)}{(\cosh(\pi x\sqrt{3}) - \cos(\pi x))(\cosh(\pi tx) - \cos(\pi tx\sqrt{3}))}. \tag{4}$$

Proof. Again, we consider the function

$$\psi(z) = \frac{\pi z \cot(\pi z)\coth(\pi t z)}{z^4 + z^2x^2 + x^4}$$

and use the theory of residues. $\psi(z)$ has simple poles at $z = n$, $z = \frac{ni}{t}$, and $z = \omega x$ for each cube root of unity, ω. The residues are computed as before. This result is due to Ramanujan. [5,Chapter 30] ■

The same method can be applied by replacing $\pi\cot(\pi z)\coth(\pi t z)$ by any function, suitably behaved as $|z| \to \infty$, with well placed poles. Several such functions, which I shall call "transform kernels" are

- $\dfrac{\pi}{\sinh(\pi t z)\sin(\pi z)}$
- $\dfrac{\pi}{\cosh(\pi t z)\cos(\pi z)}$
- $\pi\tan(\pi z)\tanh(\pi t z)$
- $\psi(z)\psi(-zt)$ where $\psi(z) = \gamma + \frac{\Gamma'(1-z)}{\Gamma(1-z)}$.

For example, consider the function

$$\varphi(z) = \frac{\pi\tan(\pi z)\tanh(\pi t z)}{z^{2m+1}}.$$

$\varphi(z)$ has simple poles at the points $z = n + \frac{1}{2}$, $z = \frac{(n+\frac{1}{2})i}{t}$ for each integer n and a pole of order $2m - 1$ at $z = 0$. Computation of the residues is routine. One finds that

$$\begin{aligned} R_{n+\frac{1}{2}}(\varphi) &= \frac{\tanh(\pi t(n + \frac{1}{2}))}{(n + \frac{1}{2})^{2m+1}}, \\ R_{\frac{(n+\frac{1}{2})i}{t}}(\varphi) &= (-1)^{m+1} t^{2m} \frac{\tanh(\frac{\pi}{t}(n + \frac{1}{2}))}{(n + \frac{1}{2})^{2m+1}}, \end{aligned}$$

while at 0, the taylor expansion of $\pi\tan(\pi z)\tanh(\pi t z)$ gives

$$R_0(\varphi) = -2(2\pi)^{2m+1}\sum_{n=1}^{m}\frac{(-1)^{n-1}(2^{2n}-1)(2^{2(m+1-n)}-1)B_{2n}B_{2(m+1-n)}}{(2n)!(2(m+1-n))!}t^{2(m-n)+1}.$$

Hence, by the theory of residues we have

$$\sum_{n=0}^{\infty}\frac{\tanh(\pi t(n+\frac{1}{2}))}{(n+\frac{1}{2})^{2m+1}} + (-1)^{m+1}t^{2m}\sum_{n=0}^{\infty}\frac{\tanh(\frac{\pi}{t}(n+\frac{1}{2}))}{(n+\frac{1}{2})^{2m+1}} \tag{5}$$

$$= (2\pi)^{2m+1}\sum_{n=1}^{m}\frac{(-1)^{n-1}(2^{2n}-1)(2^{2(m+1-n)}-1)B_{2n}B_{2(m+1-n)}}{(2n)!(2(m+1-n))!}t^{2(m-n)+1}.$$

[4,p294]

Theorem 3. *If*

$$\varphi(\alpha,\beta) = \frac{1}{2\pi^2 u^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n n\,\mathrm{csch}(\alpha n)}{n^2+\beta u^2},$$

then if $\alpha\beta = \pi^2$,

$$\varphi(\alpha,\beta) + \varphi(-\beta,-\alpha) = \frac{\pi}{\sinh(\beta\sqrt{\alpha}u)\sin(\alpha\sqrt{\beta}u)}. \tag{6}$$

Proof. We consider the function

$$\varphi(z) = \frac{\pi z\csc(z\pi)\,\mathrm{csch}(\alpha z)}{z^2+\beta u^2}$$

and use the same methods as before. Again, this is due to Ramanujan. [4,p272] ■

By considering

$$\frac{\pi z^{2m-1}}{\sin(\pi z)\sinh(\pi t z)}$$

we obtain the well known result [2,p337] that if $\alpha\beta = \pi^2$ and

$$\xi(\alpha) = \alpha^m\sum_{n=1}^{\infty}\frac{(-1)^n n^{2m-1}}{\sinh(\alpha n)},$$

then for $m > 1$

$$\xi(\alpha) = (-1)^m\xi(\beta), \tag{7}$$

and for $m = 1$, if t is real and positive,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{\sinh(\pi t n)} + t^{-2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{\sinh(\frac{\pi n}{t})} = \frac{1}{2\pi t}.$$

Recalling that

$$\int\frac{\pi n dz}{\sinh(\pi z n)} = \log(\frac{\sinh(\pi n z)}{1+\cosh(\pi n z)}),$$

we at once obtain

Theorem 4. *If*

$$f(t) = \frac{\cosh(\pi t) - 1}{\sinh(\pi t)} \frac{\cosh(2\pi t) + 1}{\sinh(2\pi t)} \frac{\cosh(3\pi t) - 1}{\sinh(3\pi t)} \cdots,$$

then

$$f(t) = t^{\frac{1}{2}} f(\frac{1}{t}). \tag{8}$$

It will be useful to keep this in mind as we shall seek out generalizations of this result (which is, if one likes, is an analogue to the transformation formula for the Dedekind Eta function) in Section 5.

Theorem 5. *Let $\phi(z) = \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n-z})$. Then for positive, real t,*

$$t^{m-1}\sum_{n=1}^{\infty}\frac{\phi(-\frac{n}{t})}{n^m} + (-1)^{m-1}\sum_{n=1}^{\infty}\frac{\phi(-nt)}{n^m} = \sum_{j=1}^{m-2}(-1)^{m-j}\zeta(j+1)\zeta(m-j)t^j. \tag{9}$$

Proof. Integrate $f(z) = \frac{\phi(z)\phi(-zt)}{z^m}$ around a square centered at the origin whose boundary does not include any poles of f, and let the sides of the square become infinite. One readily checks that the integral tends to 0. The residues at $z = n$ and $z = -\frac{n}{t}$ for n a natural number are easily calculated, while at 0, we need only expand $\phi(z)$ as a taylor series in the obvious way. An application of the residue theorem completes the proof. Note that $\phi(z) = \gamma + \frac{\Gamma'(1-z)}{\Gamma(1-z)}$ where γ is Euler's constant. ■

We have the following examples:

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^3} = \frac{7\pi^3}{2^2\,3^2\,5} \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^7} = \frac{19\pi^3}{2^2\,3^4\,5^2\,7} \tag{11}$$

$$\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{11}} = \frac{7\pi^{11}}{2\,3^5\,5^3\,7^2\,11\,13} \tag{12}$$

Of course, these are well known. [4,p293] By the Von-Stoudt Clausen Theorem, it is immediate that if we put $\sum_{n=1}^{\infty}\frac{\coth(\pi n)}{n^{4k+3}} = \pi^{4k+3}C_k$ then C_k is rational, and the denominator of C_k is devisible by all primes p less than or equal to $4k+5$.

One useful application of these evaluations is the extremely rapid computation of $\zeta(4k+3)$ for (1) with $t = 1$ can be written

$$\zeta(4k+3) + 2\sum_{n=1}^{\infty}\frac{1}{n^{4k+3}(e^{2\pi n}-1)} = -\frac{1}{2}(2\pi)^{4k+3}\sum_{n=0}^{2k+2}\frac{(-1)^n B_{2n}B_{2(2k+2-n)}}{(2n)!(2(2k+2-n))!}.$$

This particular formula is due to Ramanujan. [3,p152] Of course, we can equally as well apply this teqhnique to any of the aforementioned sums by replacing $\coth(\pi n)$, $\tanh(\pi n)$, by $1 + \dfrac{2}{e^{2\pi n}-1}$, $1 - \dfrac{2}{e^{2\pi n}+1}$, respectively. This provides a rapid means of calculation for the sum of the reciprocals of many polynomial functions evaluated at the integers.

If, in (9), $t = 1$, then

$$\sum_{n=1}^{\infty} \frac{(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1})}{n^{2k+1}} = \frac{1}{2} \sum_{j=1}^{2k-1} (-1)^j \zeta(j+1)\zeta(2k+1-j). \tag{13}$$

This result is well known, and dates back to Euler. Perhaps a less-well known result is

Theorem 6. *If*

$$\vartheta(t,x) = \sum_{n=1}^{\infty} \frac{\phi(-nt)}{n^2 - x^2},$$

then

$$t\vartheta(\frac{1}{t}, xt) - \vartheta(t,x) = \frac{1}{2x}(\phi(x)\phi(-xt) - \phi(xt)\phi(-x)). \tag{14}$$

This is obtained as above, in the obvious way. Clearly, many more results of this nature can be obtained as indicated, or by differentiating any of these with respect to any of the parameters. It is worth noting that in all cases above, we have confined the discussion to the modular transformation $t \to \frac{1}{t}$, but any general transformation $t \to \tau(t)$ will work as well. One need only consider the transform kernel

$$\pi \cot(\pi z) \coth(\pi\tau(t)z)$$

These results are most interesting when $\tau^{-1}(t)$ is simply related to $\tau(t)$. [3]

2. GENERALITIES

The previous results can be generalized. Instead of considering transform kernels of the form

$$\psi(z)\psi(itz)$$

where ψ is a function with evenly spaced poles along the real axis, we will consider kernels of the form

$$\psi(z)\psi(\zeta t_1 z)\psi(\zeta^2 t_2 z)\ldots\psi(\zeta^{n-1} t_{n-1} z)$$

where ζ is a primitive n^{th} root of unity, ψ is as above, and $t_1, t_2, \ldots t_{n-1}$ are real and positive. As an illustration, we have

Theorem 7. *If*

$$\psi(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{4m-1}}{\sinh(\pi n)(\cosh(\pi t n\sqrt{2}) - \cos(\pi t n\sqrt{2}))},$$

then if $m > 1$,

$$\psi(\frac{1}{t}) = (-1)^m t^{4m} \psi(t), \tag{15}$$

and if $m = 1$,

$$\psi(\frac{1}{t}) + t^4 \psi(t) = \frac{t^2}{8\pi^3}. \tag{16}$$

Proof. Let $\zeta = e^{\frac{\pi i}{4}}$ and consider

$$\varphi(z) = \frac{\pi z^{4m-1}}{\sin(\pi z)\sin(\pi t\zeta z)\sin(\pi \zeta^2 z)\sin(\pi t\zeta^3 z)},$$

where t is real and positive. It is easily seen that

$$\begin{aligned}\frac{1}{\sin(\pi t\zeta z)\sin(\pi t\zeta^{-1} z)} &= \frac{1}{\sin^2(\frac{\pi}{\sqrt{2}}tz)\cosh^2(\frac{\pi}{\sqrt{2}}tz) + \sinh^2(\frac{\pi}{\sqrt{2}}tz)\cos^2(\frac{\pi}{\sqrt{2}}tz)} \\ &= \frac{2}{\cosh(\pi tz\sqrt{2}) - \cos(\pi tz\sqrt{2})}.\end{aligned}$$

Now $\varphi(z)$ has simple poles at $z = n, \frac{\zeta n}{t}, \zeta^2 n, \frac{\zeta^3 n}{t}$ for each nonzero integer n, and a simple pole at $z = 0$ if $m = 1$. The residues are readily calculated:

$$R_n = -\frac{2(-1)^n n^{4m-1}}{i\sinh(\pi n)(\cosh(\pi tn\sqrt{2}) - \cos(\pi tn\sqrt{2}))}$$

$$R_{\zeta^2 n} = -\frac{2(-1)^n n^{4m-1}}{i\sinh(\pi n)(\cosh(\pi tn\sqrt{2}) - \cos(\pi tn\sqrt{2}))}$$

$$R_{\frac{\zeta n}{t}} = \frac{(-1)^m}{t^{4m}}\frac{2(-1)^n n^{4m-1}}{i\sinh(\pi n)(\cosh(\frac{\pi\sqrt{2}n}{t}) - \cos(\frac{\pi\sqrt{2}n}{t}))}$$

$$R_{\frac{\zeta^3 n}{t}} = \frac{(-1)^m}{t^{4m}}\frac{2(-1)^n n^{4m-1}}{i\sinh(\pi n)(\cosh(\frac{\pi\sqrt{2}n}{t}) - \cos(\frac{\pi\sqrt{2}n}{t}))}$$

and if $m = 1$,

$$R_0 = -\frac{1}{it^2\pi^3}.$$

It is not difficult to see that

$$\int_C \varphi(z)dz = 0,$$

where C is an infinite square. Cauchy's Theorem completes the proof. ■

As an amusing series evalution, set $t = m = 1$ in Theorem 7 to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3}{\sinh(\pi n)(\cosh(\pi n\sqrt{2}) - \cos(\pi n\sqrt{2}))} = \frac{1}{16\pi^3}. \tag{17}$$

By the same methods, we have:

Theorem 8. *If*

$$\varphi(\alpha) = \alpha^{2m}\sum_{n=0}^{\infty} \frac{(-1)^{n-1}n^{4m-1}}{\cosh(\pi(n+\frac{1}{2}))(\cosh(\alpha(n+\frac{1}{2})) - \cos(\alpha(n+\frac{1}{2})))},$$

then if $\alpha\beta = 2\pi^2$,

$$\varphi(\alpha) = (-1)^m \varphi(\beta). \tag{18}$$

As another generalization of the technique developed in section 1, consider the following transform kernel:

$$f(z) = \cot(\pi z)\cot(\pi t z e^{i\theta}).$$

The process should now be clear, and since by (3),

$$\cot(\pi t z e^{i\theta}) = \frac{\sin(2\pi t z\cos(\theta)) - i\sinh(2\pi t z\sin(\theta))}{\cosh(2\pi t z\sin(\theta)) - \cos(2\pi t z\cos(\theta))},$$

we have, after some strenuous manipulation:

Theorem 9. *If t is real, $m \geq 0$, and $0 \leq \theta \leq 2\pi$ then*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\frac{\sin(2\pi t n\cos(\theta))}{\cosh(2\pi t n\sin(\theta)) - \cos(2\pi t n\cos(\theta))}\\
&+t^{2m}\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\frac{\cos(2m\theta)\sin(\frac{2\pi n}{t}\cos(\theta)) - \sin(2m\theta)\sinh(\frac{2\pi n}{t}\cos(\theta))}{\cosh(\frac{2\pi n}{t} n\sin(\theta)) - \cos(\frac{2\pi n}{t} n\cos(\theta))}\\
=\ &\frac{1}{\pi}\zeta(2m+2)(t^{2m+1}\cos((2m+1)\theta) + t^{-1}\cos(\theta))\\
&-\frac{2}{\pi}\sum_{j=1}^{m} t^{2j-1}\cos((2j-1)\theta)\zeta(2j)\zeta(2(m+1-j)),
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&-\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\frac{\sinh(2\pi t n\cos(\theta))}{\cosh(2\pi t n\sin(\theta)) - \cos(2\pi t n\cos(\theta))}\\
&+t^{2m}\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\frac{\sin(2m\theta)\sin(\frac{2\pi n}{t}\cos(\theta)) + \cos(2m\theta)\sinh(\frac{2\pi n}{t}\cos(\theta))}{\cosh(\frac{2\pi n}{t} n\sin(\theta)) - \cos(\frac{2\pi n}{t} n\cos(\theta))}\\
=\ &\frac{1}{\pi}\zeta(2m+2)(t^{2m+1}\sin((2m+1)\theta) + t^{-1}\sin(\theta))\\
&-\frac{2}{\pi}\sum_{j=1}^{m} t^{2j-1}\sin((2j-1)\theta)\zeta(2j)\zeta(2(m+1-j)).
\end{aligned} \tag{20}$$

These somewhat cumbersome formulas are best illustrated. Let $\theta = \frac{\pi}{4}$ and $m = 1$. If $t = 1$ in (19) and (20) and add the two identities to obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\frac{\sinh(\pi n\sqrt{2}) - \sin(\pi n\sqrt{2})}{\cosh(\pi n\sqrt{2}) - \cos(\pi n\sqrt{2})} = \frac{\pi^3}{18\sqrt{2}}. \tag{21}$$

Similarly,

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\frac{\sinh(\pi n\sqrt{3})}{\cosh(\pi n\sqrt{3}) + (-1)^{n-1}} = \frac{\pi^3}{15\sqrt{3}}, \tag{22}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^{11}}\frac{\sin(\frac{\pi n}{2}(1+\sqrt{5}))}{\cos(\frac{\pi n}{2}(1+\sqrt{5})) - \cosh(\pi n(\sqrt{\frac{5-\sqrt{5}}{2}}))} = \frac{\pi^{11}}{464373000}(41 + 15\sqrt{5}). \tag{23}$$

3. Abel-Plana Analogues

Thus far, all of our techniques for transforming infinite series require that the function being summed over the integers be an even function of z. This is an unfortunate consequence of the rectangular contours employed in every case above. We shall now show how different contours can be employed to produce a variety of other results.

Theorem 10. *Let $\varphi(z)$ be an analytic function for $\Re(z)+\Im(z)\geq 0$, of at most polynomial growth. Then*

$$\int_0^\infty \frac{z\varphi(iz)}{(e^{2\pi z}-1)}dz+\int_0^\infty \frac{z\varphi(z)}{(e^{2\pi zt}-1)}dz+\int_0^{e^{-\frac{\pi i}{4}}\infty}\frac{z(\varphi(z)-\varphi(-z))}{(e^{2\pi zt}-1)(e^{2\pi iz}-1)}dz$$
$$=\frac{\varphi(0)}{4\pi t}+\sum_{n=1}^{\infty}\left\{\frac{n\varphi(\frac{ni}{t})}{t^2(e^{\frac{2\pi n}{t}}-1)}+\frac{n\varphi(n)}{(e^{2\pi nt}-1)}\right\},$$

and in particular, if $\varphi(z)=\varphi(-z)$ then

$$\int_0^\infty \frac{z\varphi(iz)}{(e^{2\pi z}-1)}dz+\int_0^\infty \frac{z\varphi(z)}{(e^{2\pi zt}-1)}dz=\frac{\varphi(0)}{4\pi t}+\sum_{n=1}^{\infty}\left\{\frac{n\varphi(\frac{ni}{t})}{t^2(e^{\frac{2\pi n}{t}}-1)}+\frac{n\varphi(n)}{(e^{2\pi nt}-1)}\right\}. \quad (24)$$

Proof. Let γ_1, γ_2, γ_3 be the contours below (Figure 1) in the first, second, and fourth quadrants respectively, orientated counterclockwise. Let ι_n^+ be the semicircular

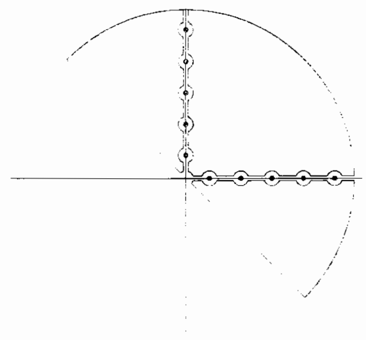


Figure 1:

indent of radius ϵ centered at $\frac{ni}{t}$ in the first quadrant, ι_n^- the analogous indent in the second. Similarly, ρ_n^+ is the semicircular indent of radius ϵ centered at n in the first quadrant while ρ_n^- is the semicircular indent of radius ϵ centered at n in the fourth quadrant. Consider the functions

$$\Psi_1(z)=\frac{z\varphi(z)}{(e^{2\pi zt}-1)(e^{-2\pi iz}-1)}$$
$$\Psi_2(z)=\frac{z\varphi(z)}{(e^{-2\pi zt}-1)(e^{-2\pi iz}-1)}$$
$$\Psi_3(z)=\frac{z\varphi(z)}{(e^{2\pi zt}-1)(e^{2\pi iz}-1)}$$

where t is real and positive. It is clear that $\Psi_i(z)$ has simple poles at $z=\frac{ni}{t}$ and $z=n$ for each nonzero integer n, and a simple pole at 0, for each i. Hence,

$$\lim_{\epsilon\to 0}\int_{\iota_n^+}\Psi_1(z)dz = \lim_{\epsilon\to 0}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{(\frac{ni}{t}+\varepsilon e^{i\theta})\varphi(\frac{ni}{t}+\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta}{(e^{2\pi t(\frac{ni}{t}+\varepsilon e^{i\theta})}-1)(e^{-2\pi i(\frac{ni}{t}+\varepsilon e^{i\theta})}-1)}=\frac{n\varphi(\frac{ni}{t})}{2t^2(e^{\frac{2\pi n}{t}}-1)}$$

$$\lim_{\epsilon\to 0}\int_{\iota_n^-}\Psi_2(z)dz = \lim_{\epsilon\to 0}\int_{-\frac{\pi}{2}}^{-\frac{3\pi}{2}}\frac{(\frac{ni}{t}+\varepsilon e^{i\theta})\varphi(\frac{ni}{t}+\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta}{(e^{-2\pi t(\frac{ni}{t}+\varepsilon e^{i\theta})}-1)(e^{-2\pi i(\frac{ni}{t}+\varepsilon e^{i\theta})}-1)}=-\frac{n\varphi(\frac{ni}{t})}{2t^2(e^{\frac{2\pi n}{t}}-1)}$$

$$\lim_{\epsilon\to 0}\int_{\rho_n^+}\Psi_1(z)dz = \lim_{\epsilon\to 0}\int_{\pi}^{0}\frac{(n+\varepsilon e^{i\theta})\varphi(n+\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta}{(e^{2\pi t(n+\varepsilon e^{i\theta})}-1)(e^{-2\pi i(n+\varepsilon e^{i\theta})}-1)}=\frac{n\varphi(n)}{2(e^{2\pi nt}-1)}$$

$$\lim_{\epsilon\to 0}\int_{\rho_n^-}\Psi_3(z)dz = \lim_{\epsilon\to 0}\int_{0}^{-\pi}\frac{(n+\varepsilon e^{i\theta})\varphi(n+\varepsilon e^{i\theta})i\varepsilon e^{i\theta}d\theta}{(e^{2\pi t(n+\varepsilon e^{i\theta})}-1)(e^{2\pi i(n+\varepsilon e^{i\theta})}-1)}=-\frac{n\varphi(n)}{2(e^{2\pi nt}-1)}$$

$$\lim_{\epsilon\to 0}\int_{\rho_0^+}\Psi_1(z)dz = \lim_{\epsilon\to 0}\int_{\frac{\pi}{2}}^{0}\frac{\varphi(\varepsilon e^{i\theta})i\varepsilon^2 e^{2i\theta}d\theta}{(e^{2\pi t\varepsilon e^{i\theta}}-1)(e^{-2\pi i\varepsilon e^{i\theta}}-1)}=\frac{\varphi(0)}{8\pi t}$$

$$\lim_{\epsilon\to 0}\int_{\rho_0^-}\Psi_3(z)dz = \lim_{\epsilon\to 0}\int_{0}^{-\frac{\pi}{4}}\frac{\varphi(\varepsilon e^{i\theta})i\varepsilon^2 e^{2i\theta}d\theta}{(e^{2\pi t\varepsilon e^{i\theta}}-1)(e^{2\pi i\varepsilon e^{i\theta}}-1)}=-\frac{\varphi(0)}{16\pi t}$$

$$\lim_{\epsilon\to 0}\int_{\iota_0^-}\Psi_2(z)dz = \lim_{\epsilon\to 0}\int_{\frac{3\pi}{4}}^{\frac{\pi}{2}}\frac{\varphi(\varepsilon e^{i\theta})i\varepsilon^2 e^{2i\theta}d\theta}{(e^{-2\pi t\varepsilon e^{i\theta}}-1)(e^{-2\pi i\varepsilon e^{i\theta}}-1)}=-\frac{\varphi(0)}{16\pi t}$$

Since the integrals along the segments of the large circular arc tend to 0 as $R\to\infty$, (as is easily checked by estimating the integrand on the arc), as $\epsilon\to 0$ and $R\to\infty$ we have

$$\int_{\gamma_1}\Psi_1(z)dz-\int_{\gamma_2}\Psi_2(z)dz-\int_{\gamma_3}\Psi_3(z)dz=\frac{\varphi(0)}{4\pi t}+\sum_{n=1}^{\infty}\left\{\frac{n\varphi(\frac{ni}{t})}{t^2(e^{\frac{2\pi n}{t}}-1)}+\frac{n\varphi(n)}{(e^{2\pi nt}-1)}\right\}$$
$$-\int_0^{i\infty}(\Psi_1(z)+\Psi_2(z))\,dz+\int_0^{\infty}(\Psi_1(z)+\Psi_3(z))\,dz$$
$$+\int_0^{-e^{-\frac{\pi i}{4}}\infty}\Psi_2(z)dz-\int_0^{e^{-\frac{\pi i}{4}}\infty}\Psi_3(z)dz.$$

But by Cauchy's Theorem, (since Ψ_1, Ψ_2, Ψ_3 are analytic inside of γ_1, γ_2, γ_3)

$$\int_{\gamma_1}\Psi_1(z)dz-\int_{\gamma_2}\Psi_2(z)dz-\int_{\gamma_3}\Psi_3(z)dz=0$$

■

Theorem 10 has many interesting applications and generalizes many of the results in [6,Chapter 37]. For instance, putting $\varphi(z)=z^{2m}$, $m\geq 1$ and recalling that

$$\int_0^{\infty}\frac{z^{2m+1}}{(e^{2\pi zt}-1)}dz=\sum_{n=1}^{\infty}\int_0^{\infty}z^{2m+1}e^{-2\pi nzt}dz=\frac{(2m+1)!}{(2\pi t)^{2m+2}}\zeta(2m+2)=\frac{(-1)^m}{t^{2m+2}}\frac{B_{2m+2}}{4m+2},$$

it follows immediately that [3,Proposition 2.6]

$$\frac{(-1)^m}{t^{2m+2}}\sum_{n=1}^{\infty}\frac{n^{2m+1}}{(e^{\frac{2\pi n}{t}}-1)}+\sum_{n=1}^{\infty}\frac{n^{2m+1}}{(e^{2\pi nt}-1)}=\left(1+\frac{(-1)^m}{t^{2m+2}}\right)\frac{B_{2m+2}}{4m+2}. \qquad (25)$$

This is the classical Eisenstein Series transformation, and is well known. As an example, set $t = 1$ and $m = 2k$ in (25) to obtain

$$\sum_{n=1}^{\infty} \frac{n^{4k+1}}{(e^{2\pi n} - 1)} = \frac{B_{4k+2}}{8k+2}. \tag{26}$$

Now set $\varphi(z) = 1$ in (24) to obtain

$$\frac{1}{4\pi t} + \sum_{n=1}^{\infty} \left\{ \frac{n}{t^2(e^{\frac{2\pi n}{t}} - 1)} + \frac{n}{(e^{2\pi nt} - 1)} \right\} = \frac{1}{24t^2} + \frac{1}{24}.$$

Integrating this with respect to t then gives

$$\frac{\log(t)}{4\pi} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \log(1 - e^{-2\pi nt}) - \frac{1}{2\pi} \sum_{n=1}^{\infty} \log(1 - e^{-\frac{2\pi n}{t}}) = \frac{1}{24}(t - \frac{1}{t}).$$

(since the constant of integration is obviously 0). Hence, if $\alpha\beta = 4\pi^2$ then

$$\alpha^6 e^{-\alpha} \prod_{n=1}^{\infty} (1 - e^{-n\alpha})^{24} = \beta^6 e^{-\beta} \prod_{n=1}^{\infty} (1 - e^{-n\beta})^{24}, \tag{27}$$

which is the transformation formula for the classical Dedekind Eta function. [4,p256]

By exactly the same methods as before, we have

Theorem 11. *If $\varphi(z)$ is an anylytic function for $\Re(z) + \Im(z) \geq 0$ of polynomial growth, and if $\varphi(-z) = -\varphi(z)$ then*

$$2\sum_{n=0}^{\infty} \left(\frac{\varphi(2n+1)}{e^{(2n+1)\pi t} + 1} - \frac{i}{t} \frac{\varphi((2n+1)\frac{i}{t})}{e^{(2n+1)\frac{\pi}{t}} + 1} \right) = \int_0^{\infty} \frac{\varphi(z)}{e^{\pi tz} + 1} dz - i \int_0^{\infty} \frac{\varphi(iz)}{e^{\pi z} + 1}. \tag{28}$$

Hence, as before, if $\alpha\beta = \pi^2$ then

$$\alpha^{2k} \sum_{n=0}^{\infty} \frac{(2n+1)^{4k-1}}{e^{(2n+1)\alpha} + 1} - \beta^{2k} \sum_{n=0}^{\infty} \frac{(2n+1)^{4k-1}}{e^{(2n+1)\beta} + 1} = \frac{2^{4k-1} - 1}{8k} B_{4k}(\alpha^{2k} - \beta^{2k}), \tag{29}$$

and

$$\alpha^{2k} \sum_{n=0}^{\infty} \frac{(2n+1)^{4k+1}}{e^{(2n+1)\alpha} + 1} + \beta^{2k} \sum_{n=0}^{\infty} \frac{(2n+1)^{4k+1}}{e^{(2n+1)\beta} + 1} = \frac{2^{4k+1} - 1}{8k} B_{4k+2}(\alpha^{2k+1} + \beta^{2k+1}). \tag{30}$$

Now

$$\int_0^{\infty} \frac{\sin(az)}{e^{\pi tz} + 1} dz = \sum_{n=1}^{\infty} \int_0^{\infty} (-1)^{n-1} e^{-\pi ntz} \sin(az) dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\pi nt}{a^2 + (\pi nt)^2} = \frac{1}{2a} - \frac{1}{2t\sinh(\frac{a}{t})},$$

whence if α, β are real and positive with $\alpha < \pi$, Theorem 11 gives

$$\alpha \sum_{n=0}^{\infty} \frac{\sin((2n+1)\alpha)}{e^{(2n+1)\pi\frac{\alpha}{\beta}} + 1} + \beta \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\beta)}{e^{(2n+1)\pi\frac{\beta}{\alpha}} + 1} = \frac{1}{4} \left(\frac{\alpha}{\sin(\alpha)} - \frac{\beta}{\sinh(\beta)} \right). \tag{31}$$

Let u, v be real and positive, with $0 < \theta < \pi$. Suppose that $\varphi(z)$ is analytic of polynomial growth with $\varphi(-z) = \varphi(z)$. By considering the functions

$$\begin{aligned}\Phi_1(z) &= \frac{z\varphi(z)}{(e^{-2\pi izv}-1)(e^{2\pi izue^{-i\theta}}-1)}\\ \Phi_2(z) &= \frac{z\varphi(z)}{(e^{-2\pi izv}-1)(e^{-2\pi izue^{-i\theta}}-1)}\\ \Phi_3(z) &= \frac{z\varphi(z)}{(e^{2\pi izv}-1)(e^{2\pi izue^{-i\theta}}-1)}\end{aligned}$$

integrated around a contour very similar to that of before, it can be shown (just as above) that

Theorem 12. *If $\varphi(z)$ is an anylytic function for $\Re(z)+\Im(z)\geq 0$ of polynomial growth, and if $\varphi(-z) = -\varphi(z)$ then for u, v real and positive, and $0<\theta<\pi$,*

$$\frac{e^{i\theta}}{i}\frac{\varphi(0)}{4\pi uv}+\frac{1}{v^2}\sum_{n=1}^{\infty}\frac{n\varphi(\frac{n}{v})}{e^{2\pi in\frac{u}{v}\cos(\theta)}e^{2\pi n\frac{u}{v}\sin(\theta)}-1}-\frac{e^{2i\theta}}{u^2}\sum_{n=1}^{\infty}\frac{n\varphi(\frac{ne^{i\theta}}{u})}{e^{-2\pi in\frac{v}{u}\cos(\theta)}e^{2\pi n\frac{v}{u}\sin(\theta)}-1}$$

$$= \int_0^{\infty}\frac{z\varphi(z)}{e^{2\pi iuze^{-i\theta}}-1}dz - e^{2i\theta}\int_0^{\infty}\frac{z\varphi(ze^{i\theta})}{e^{-2\pi ivze^{i\theta}}-1}dz. \quad (32)$$

By applying this result in the obvious ways, we deduce, for $m \geq 1$, the following examples (See, for instance,[3,p157]):

$$\sum_{n=1}^{\infty}\frac{n^{2m+1}}{(-1)^n e^{\pi n\sqrt{3}}-1} = \sum_{n=1}^{\infty}\frac{n^{2m+1}}{(-1)^n e^{\frac{\pi n}{\sqrt{3}}}-1} = \frac{B_{2m+2}}{4m+4}, \quad (33)$$

$$\sum_{n\equiv\pm1(3)}\frac{n^{2m+1}}{(-1)^n e^{\frac{\pi n}{\sqrt{3}}}-1} = \frac{B_{2m+2}}{4m+4}(1-3^{2m+1}), \quad (34)$$

$$\sum_{n=1}^{\infty}\frac{n}{(-1)^n e^{\pi n\sqrt{3}}-1} = \frac{1}{24}-\frac{1}{4\pi\sqrt{3}}, \quad (35)$$

$$\sum_{n=1}^{\infty}\frac{n}{(-1)^n e^{\frac{\pi n}{\sqrt{3}}}-1} = \frac{1}{24}-\frac{\sqrt{3}}{4\pi}, \quad (36)$$

$$\sum_{n=1}^{\infty}\frac{n}{(-1)^n e^{\pi n}-1} = \frac{1}{24}-\frac{1}{4\pi}. \quad (37)$$

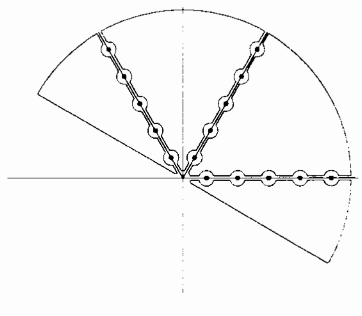
Theorem 13. *If u, v are real and positive and $\varphi(z)$ is an anylytic function of polynomial growth, and if $\varphi(-z) = -\varphi(z)$ then*

$$\sum_{n=1}^{\infty}\frac{n^2\varphi(n)}{(e^{2\pi iune^{-\frac{2\pi i}{3}}}-1)(e^{2\pi ivne^{-\frac{\pi i}{3}}}-1)}+\frac{1}{v^3}\sum_{n=1}^{\infty}\frac{n^2\varphi(\frac{ne^{\frac{\pi i}{3}}}{v})}{(e^{2\pi in\frac{1}{v}e^{-\frac{2\pi i}{3}}}-1)(e^{2\pi in\frac{u}{v}e^{-\frac{\pi i}{3}}}-1)}$$

$$+\frac{1}{u^3}\sum_{n=1}^{\infty}\frac{n^2\varphi(\frac{ne^{\frac{2\pi i}{3}}}{u})}{(e^{2\pi in\frac{v}{u}e^{-\frac{2\pi i}{3}}}-1)(e^{2\pi in\frac{1}{u}e^{-\frac{\pi i}{3}}}-1)}+\frac{\varphi(0)}{8\pi^2 uv}$$

$$= \int_0^\infty \frac{z^2\varphi(z)dz}{(e^{2\pi iuze^{-\frac{2\pi i}{3}}}-1)(e^{2\pi ivze^{-\frac{\pi i}{3}}}-1)} + \int_0^\infty \frac{z^2\varphi(ze^{\frac{2\pi i}{3}})dz}{(e^{2\pi ivze^{-\frac{2\pi i}{3}}}-1)(e^{2\pi ize^{-\frac{\pi i}{3}}}-1)}$$
$$+ \int_0^\infty \frac{z^2\varphi(ze^{\frac{\pi i}{3}})dz}{(e^{2\pi ize^{-\frac{2\pi i}{3}}}-1)(e^{2\pi iuze^{-\frac{\pi i}{3}}}-1)}. \quad (38)$$

Proof. Let u, v be real and positive. Let γ_1, γ_2, γ_3, γ_4, be the contours below (Figure two) oriented counterclockwise with γ_1 bounding the region given in polar coordinates by $0 \leq \theta < \frac{\pi}{3}$, γ_2, the region $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}$, γ_3, the region $\frac{2\pi}{3} \leq \theta < \frac{5\pi}{6}$, and γ_4, the region $-\frac{\pi}{6} \leq \theta < 0$.



Let $\varphi(z)$ be analytic with $\varphi(z) = \varphi(-z)$, and consider the functions

$$\Psi_1(z) = \frac{z^2\varphi(z)}{(e^{-2\pi iz}-1)(e^{2\pi iuze^{-\frac{2\pi i}{3}}}-1)(e^{2\pi ivze^{-\frac{\pi i}{3}}}-1)}$$
$$\Psi_2(z) = \frac{z^2\varphi(z)}{(e^{-2\pi iz}-1)(e^{2\pi iuze^{-\frac{2\pi i}{3}}}-1)(e^{-2\pi ivze^{-\frac{\pi i}{3}}}-1)}$$
$$\Psi_3(z) = \frac{z^2\varphi(z)}{(e^{-2\pi iz}-1)(e^{-2\pi iuze^{-\frac{2\pi i}{3}}}-1)(e^{-2\pi ivze^{-\frac{\pi i}{3}}}-1)}$$
$$\Psi_4(z) = \frac{z^2\varphi(z)}{(e^{2\pi iz}-1)(e^{2\pi iuze^{-\frac{2\pi i}{3}}}-1)(e^{2\pi ivze^{-\frac{\pi i}{3}}}-1)}.$$

Just as before, we compute

$$\int_{\gamma_1} \Psi_1(z)dz - \int_{\gamma_2} \Psi_2(z)dz + \int_{\gamma_3} \Psi_3(z)dz - \int_{\gamma_4} \Psi_4(z)dz,$$

by the residue theorem and let the radius of the large arc tend to infinity while the radii of the small indents tend to zero. The result follows. ■

4. Fourier Transform Methods

It is well known that $e^{\pi iz^2}$ is its own fourier transform. Poisson summation may be used with this fact to prove the classical transformation formula for $\vartheta(\tau) = 1 + 2\sum_{n=1}^\infty e^{i\pi\tau n^2}$. The Mellin transform of this modular form in turn gives the functional equation for the Riemann Zeta function. Analogously, any function that is its own fourier transform (henceforth "self-reciprocal") leads to a modular form via the Poisson Summation formula, which leads (by Mellin Transformation) to a functional equation for some Dirichlet

Series. In this section, we will investigate a wide class of self-reciprocal functions and the corresponding modular forms and functional equations.

Theorem 14. *Let f be any function defined on the integers, with period m. Put*

$$c(z) = \sum_{n=1}^{m-1} f(n) e^{2\pi n z},$$

and let

$$\hat{f}(x) = c(\frac{ix}{m}).$$

If $\alpha\beta = \frac{2\pi}{m}$ then

$$\frac{1}{\alpha}\int_{-\infty}^{\infty} \frac{\sum_{n=1}^{m-1} f(n)e^{\beta n u}}{e^{\beta m u}-1} e^{-ixu} du = -i(\frac{1}{2}\hat{f}(0) + \frac{\sum_{k=1}^{m-1}\hat{f}(k)e^{\alpha k x}}{e^{\alpha m x}-1}). \qquad (39)$$

Proof. With the above notation, let R be real and consider

$$\int_{C_R} \frac{c(z)}{e^{2\pi m z}-1} e^{-ixz} dz$$

where C_R is a rectangle oriented counterclockwise with horizontal sides on the lines $\Im(z) = 1$ and $\Im(z) = 0$,vertical sides passing through the points $\pm R$, and semicircular indents at $z = 0$ and $z = i$. It is easily seen that as $R \to \infty$, the integrals along the vertical sides vanish. The integrand has poles at $\frac{ki}{m}$ for $1 \le k \le m-1$ with residue

$$\frac{\sum_{n=1}^{m-1} f(n)e^{\frac{2\pi n k i}{m}}}{2\pi m} e^{\frac{kx}{m}} = \frac{1}{2\pi m}\hat{f}(k)e^{\frac{kx}{m}}.$$

Now,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_0^{-\pi} \frac{\sum_{n=1}^{m-1} f(n)e^{2\pi n(i+\varepsilon e^{i\theta})}}{e^{2\pi m(i+\varepsilon e^{i\theta})}-1} e^{-ix(i+\varepsilon e^{i\theta})} i\varepsilon e^{i\theta} d\theta &= -\frac{i}{2m}e^x \hat{f}(0), \\
\lim_{\varepsilon\to 0}\int_{-\pi}^{0} \frac{\sum_{n=1}^{m-1} f(n)e^{2\pi n\varepsilon e^{i\theta}}}{e^{2\pi m\varepsilon e^{i\theta}}-1} e^{-ix\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta &= \frac{i}{2m}\hat{f}(0),
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{C_\infty} \frac{c(z)}{e^{2\pi m z}-1} e^{-ixz} dz = \int_{-\infty}^{\infty} \frac{c(u)}{e^{2\pi m u}-1} e^{-ixu} du - e^x \int_{-\infty}^{\infty} \frac{c(u)}{e^{2\pi m u}-1} e^{-ixu} du \\
&+ \lim_{\varepsilon\to 0}\int_0^{-\pi} \frac{c(i+\varepsilon e^{i\theta})}{e^{2\pi m(i+\varepsilon e^{i\theta})}-1} e^{-ix(i+\varepsilon e^{i\theta})} i\varepsilon e^{i\theta} d\theta + \lim_{\varepsilon\to 0}\int_{-\pi}^{0} \frac{c(\varepsilon e^{i\theta})}{e^{2\pi m\varepsilon e^{i\theta}}-1} e^{-ix\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta \\
= \;& (1-e^x)(\frac{i}{2m}\hat{f}(0) + \int_{-\infty}^{\infty} \frac{c(u)}{e^{2\pi m u}-1} e^{-ixu} du).
\end{aligned}$$

Thus, by Cauchy's Theorem,

$$\int_{C_\infty} \frac{c(z)}{e^{2\pi m z}-1} e^{-ixz} dz = \frac{i}{m}\sum_{k=1}^{m-1} \hat{f}(k)e^{\frac{kx}{m}}.$$

It follows that

$$\int_{-\infty}^{\infty}\frac{\sum_{n=1}^{m-1}f(n)e^{2\pi nu}}{e^{2\pi mu}-1}e^{-ixu}du=-\frac{i}{m}(\frac{1}{2}\hat{f}(0)+\frac{\sum_{k=1}^{m-1}\hat{f}(k)e^{\frac{kx}{m}}}{e^{x}-1}).$$

■

Note that for every $f(n)$ that is its own finite fourier transform (up to constant multiplication), Theorem 14 gives a corresponding $\varphi(z)$ that is its own continuous fourier transform.

As noted above, we can take mellin transforms of both sides of Theorem 14. Suppose, for simplicity, that $\hat{f}(0)=0$. Thus, with α and β strictly positive and $\alpha\beta=\pi^2$as before, and $\Re(s)>0$,

$$\begin{aligned}
&\int_0^{\infty}x^{s-1}\frac{\sum_{k=1}^{m-1}\hat{f}(k)e^{\alpha kx}}{e^{\alpha mx}-1}dx=\frac{i}{\alpha}\int_0^{\infty}\int_{-\infty}^{\infty}x^{s-1}\frac{\sum_{n=1}^{m-1}f(n)e^{\beta nu}}{e^{\beta mu}-1}e^{-ixu}dudx\\
=&\ \frac{i}{\alpha}\int_0^{\infty}\int_0^{\infty}x^{s-1}\frac{\sum_{n=1}^{m-1}f(n)(e^{\beta nu}e^{-ixu}-e^{\beta(m-n)u}e^{ixu})}{e^{\beta mu}-1}dxdu\\
=&\ \frac{\Gamma(s)}{\alpha(i)^{s-1}}\int_0^{\infty}u^{-s}\frac{\sum_{n=1}^{m-1}f(n)(e^{\beta nu}-e^{-\pi is}e^{\beta(m-n)u})}{e^{\beta mu}-1}du\\
=&\ \frac{\Gamma(s)}{\alpha(i)^{s-1}}\sum_{j=1}^{\infty}\sum_{n=1}^{m-1}f(n)\int_0^{\infty}u^{-s}(e^{-\beta u(mj-n)}-e^{-\pi is}e^{-\beta u(mj-m+n)})du\\
=&\ \frac{\Gamma(s)\Gamma(1-s)}{\alpha\beta^{1-s}(i)^{s-1}}\sum_{j=1}^{\infty}\sum_{n=1}^{m-1}f(n)(\frac{1}{(mj-n)^{1-s}}-\frac{e^{-\pi is}}{(m(j-1)+n)^{1-s}})\\
=&\ \frac{\Gamma(s)\Gamma(1-s)}{\alpha\beta^{1-s}(i)^{s-1}}\sum_{j=-\infty}^{\infty}\sum_{n=1}^{m-1}\frac{f(n)}{(mj-n)^{1-s}}.
\end{aligned}$$

All of the changes in order of integration and summation are readily justified by absolute convergence. Also,

$$\begin{aligned}
\int_0^{\infty}x^{s-1}\frac{\sum_{k=1}^{m-1}\hat{f}(k)e^{\alpha kx}}{e^{\alpha mx}-1}dx &= \sum_{j=1}^{\infty}\sum_{k=1}^{m-1}\hat{f}(k)\int_0^{\infty}x^{s-1}e^{-\alpha x(mj-k)}dx\\
&= \frac{\Gamma(s)}{\alpha^s}\sum_{j=1}^{\infty}\sum_{k=1}^{m-1}\frac{\hat{f}(k)}{(mj-k)^s}
\end{aligned}$$

where the change in order of summation and integration is legitimate as before. Thus, we have

Theorem 15. *If $f(n)$ is a function on the integers with period m, and $f(0)=0$, and if $\hat{f}(z)=\sum_{n=0}^{m-1}f(n)e^{\frac{2\pi inz}{m}}$, $\hat{f}(0)=0$, then for $\Re(s)>0$,*

$$\sum_{j=1}^{\infty}\sum_{k=1}^{m-1}\frac{\hat{f}(k)}{(mj-k)^s}=(\frac{2\pi}{m})^{s-1}\Gamma(1-s)e^{\frac{\pi i}{2}(1-s)}\sum_{j=-\infty}^{\infty}\sum_{n=1}^{m-1}\frac{f(n)}{(mj-n)^{1-s}}. \tag{40}$$

For a different proof of this, see [1]. As an illustration, let we will set $f=\chi$, any primitive Dirichlet character modulo m. That is, χ satisfies

1. $\chi(kn) = \chi(k)\chi(n)$ for all k, n.
2. $\chi(k) = 0$ whenever $(k, m) \neq 1$.
3. $\hat{\chi}(k) = \sum_{(n,m)=1} \chi(n) e^{\frac{2\pi n k i}{m}} = 0$ whenever $(k, m) \neq 1$.

We have

$$\hat{\chi}(k) = \sum_{(n,m)=1} \chi(n) e^{\frac{2\pi n k i}{m}} = \bar{\chi}(k) \sum_{(n,m)=1} \chi(n) e^{\frac{2\pi n i}{m}} = \bar{\chi}(k) G(1, \chi), \tag{41}$$

where $G(1, \chi) = c(\frac{i}{m})$ is Gauss' sum modulo m. By theorem 15, one finds

$$\chi(-1)G(1,\chi)\sum_{j=1}^{\infty}\sum_{k=1}^{m-1}\frac{\bar{\chi}(mj-k)}{(mj-k)^s} = (\frac{2\pi}{m})^{s-1}\Gamma(1-s)e^{\frac{\pi i}{2}(1-s)}\sum_{j=-\infty}^{\infty}\sum_{n=1}^{m-1}\frac{\chi(n)}{(mj-n)^{1-s}}.$$

Changing s to $1-s$, we have

$$\begin{aligned}\chi(-1)G(1,\chi)L(1-s,\bar{\chi}) &= (\frac{m}{2\pi})^s\Gamma(s)e^{\frac{\pi i s}{2}}\sum_{j=1}^{\infty}\sum_{n=1}^{m-1}(\frac{\chi(n)}{(mj-n)^s} + \frac{e^{-\pi i s}\chi(n)}{(mj-m+n)^s}) \\ &= (\frac{m}{2\pi})^s\Gamma(s)e^{\frac{\pi i s}{2}}(\chi(-1)+e^{-\pi i s})L(s,\chi).\end{aligned}$$

Now, replacing χ with $\bar{\chi}$, multiplying both sides by $G(1,\chi)$, and recalling that

$$\chi(-1)G(1,\chi)G(1,\bar{\chi}) = m$$

it follows that

$$L(1-s,\chi) = \frac{m^{s-1}\Gamma(s)}{(2\pi)^s}(\chi(-1)e^{\frac{\pi i s}{2}} + e^{-\frac{\pi i s}{2}})L(s,\bar{\chi}) \tag{42}$$

Theorem 16. *Let*

$$\hat{f}(n) = \sum_{n=1}^{m} f(n) e^{\frac{(2n-1)(2k-1)}{2m}\pi i}.$$

If $\alpha\beta = \frac{2\pi}{m}$, *then*

$$\frac{1}{\alpha}\int_{-\infty}^{\infty}\frac{\sum_{n=1}^{m} f(n)e^{(n-\frac{1}{2})\beta z}}{e^{m\beta z}+1}e^{-ixz}dz = -i\frac{\sum_{n=1}^{m}\hat{f}(n)e^{(n-\frac{1}{2})\alpha z}}{e^{m\alpha z}+1} \tag{43}$$

Proof. Integrate the function

$$\varphi(z) = \frac{\sum_{n=1}^{m} f(n)e^{(2n-1)\pi z}}{e^{2\pi m z}+1}e^{-ixz}$$

around the same contour as above, without the indents. Again, it is readily verified that the integrals along the vertical sides are 0 and that $\varphi(z+i) = -\varphi(z)$, so that, as before,

$$\int_{C_\infty}\varphi(z)dz = \int_{-\infty}^{\infty}\varphi(z)dz + e^x\int_{-\infty}^{\infty}\varphi(z)dz.$$

$\varphi(z)$ has a simple pole in C_∞ at $z = \frac{(2k-1)i}{2m}$ for $1 \le k \le m$ with residue

$$-\frac{\sum_{n=1}^{m} f(n) e^{\frac{(2n-1)(2k-1)}{2m}\pi i}}{2\pi m i} e^{\frac{(2k-1)}{2m}x},$$

whence by Cauchy's theorem and some elementary manipulation, the result follows. ∎

Any function f that satisfies

$$f(k) = p \sum_{n=1}^{m} f(n) e^{\frac{(2n-1)(2k-1)}{2m}\pi i} \tag{44}$$

will produce a function that is its own continuous fourier transform up to a constant factor. We show that such $f(n)$ are easily found.

Let F^m denote the $m \times m$ matrix given by

$$F^m_{kj} = e^{\frac{(2k-1)(2j-1)}{2m}\pi i}$$

for $1 \le k, j \le m$. Then it is not difficult to see that finding functions on the first m natural numbers that satisfy (44) corresponds to computing eigenvectors of F^m. It is obvious that F^m is a symmetric matrix, so that

$$(F^m)^2_{kj} = \sum_{n=1}^{m} e^{\frac{(2k-1)(2n-1)}{2m}\pi i} e^{\frac{(2n-1)(2j-1)}{2m}\pi i} = \sum_{n=1}^{m} e^{\frac{(k+j-1)(2n-1)}{m}\pi i} = m\delta_{(k+j-1)m},$$

where δ_{kj} is the Kronecker delta function. Hence,

$$(F^m)^4_{kj} = m^2 \sum_{n=1}^{m} \delta_{(k+n-1)m} \delta_{(n+j-1)m} = m^2,$$

so that

$$(F^m)^4 = m^2 I, \tag{45}$$

where I is the $m \times m$ identity matrix, from which it readily follows that all the eigenvalues of F^m are in the set

$$S_m = \{\sqrt{m}, -\sqrt{m}, i\sqrt{m}, -i\sqrt{m}\}$$

Proposition 17. *For $m = 3$, the eigenvectors of F^m are*

$$\{1, -1 + \sqrt{3}, 1\},$$
$$\{-1, 0, 1\},$$
$$\{1, -1 - \sqrt{3}, 1\},$$

with eigenvalues $i\sqrt{3}$, $\sqrt{3}$, $-i\sqrt{3}$, respectively. For $m = 4$, the eigenvectors are

$$\{-1, 1 + \sqrt{2} + \sqrt{2(2+\sqrt{2})}, -1 - \sqrt{2} - \sqrt{2(2+\sqrt{2})}, 1\},$$
$$\{1, 1 - \sqrt{2} - \sqrt{2(2-\sqrt{2})}, 1 - \sqrt{2} - \sqrt{2(2-\sqrt{2})}, 1\},$$
$$\{1, 1 - \sqrt{2} + \sqrt{2(2-\sqrt{2})}, 1 - \sqrt{2} + \sqrt{2(2-\sqrt{2})}, 1\},$$
$$\{-1, 1 + \sqrt{2} - \sqrt{2(2+\sqrt{2})}, -1 - \sqrt{2} + \sqrt{2(2+\sqrt{2})}, 1\},$$

with eigenvalues -2, $-2i$, $2i$, 2, *respectively. For* $m = 5$, *the eigenvectors are*

$$\{-1, \frac{1}{2}(1+\sqrt{5}+\sqrt{2(5+\sqrt{5})}), 0, -\frac{1}{2}(1+\sqrt{5}+\sqrt{2(5+\sqrt{5})}), 1\},$$
$$\{1, -1, 1-\sqrt{5}, -1, 1\},$$
$$\{1, 0, \frac{1}{2}(1+\sqrt{5}), 0, 1\},$$
$$\{0, 1, -\frac{1}{2}(1+\sqrt{5}), 1, 0\},$$
$$\{-1, \frac{1}{2}(1+\sqrt{5}-\sqrt{2(5+\sqrt{5})}), 0, -\frac{1}{2}(1+\sqrt{5}-\sqrt{2(5+\sqrt{5})}), 1\},$$

with eigenvalues $-\sqrt{5}$, $-i\sqrt{5}$, $i\sqrt{5}$, $i\sqrt{5}$, $\sqrt{5}$, *respectively.*

From these computations, and Theorem 16, we at once have

Proposition 18. *If* $\alpha\beta = \frac{2\pi}{3}$ *then*

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(\beta z) - (\frac{1+\sqrt{3}}{2})}{\cosh(\frac{3}{2}\beta z)}\cos(xz)dz = -\sqrt{\alpha}\frac{\cosh(\alpha x) - (\frac{1+\sqrt{3}}{2})}{\cosh(\frac{3}{2}\alpha x)}$$

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(\beta z) + (\frac{\sqrt{3}-1}{2})}{\cosh(\frac{3}{2}\beta z)}\cos(xz)dz &= \sqrt{\alpha}\frac{\cosh(\alpha x) + (\frac{\sqrt{3}-1}{2})}{\cosh(\frac{3}{2}\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\sinh(\beta z)}{\cosh(\frac{3}{2}\beta z)}\sin(xz)dz &= \sqrt{\alpha}\frac{\sinh(\alpha x)}{\cosh(\frac{3}{2}\alpha x)}. \qquad (46)
\end{aligned}$$

If $\alpha\beta = \frac{\pi}{2}$ *then*

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(\frac{3}{2}\beta z) + \sigma\cosh(\frac{1}{2}\beta z)}{\cosh(2\beta z)}\cos(xz)dz &= -\sqrt{\alpha}\frac{\cosh(\frac{3}{2}\alpha x) + \sigma\cosh(\frac{1}{2}\alpha x)}{\cosh(2\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(\frac{3}{2}\beta z) + \varepsilon\cosh(\frac{1}{2}\beta z)}{\cosh(2\beta z)}\cos(xz)dz &= \sqrt{\alpha}\frac{\cosh(\frac{3}{2}\alpha x) + \varepsilon\cosh(\frac{1}{2}\alpha x)}{\cosh(2\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\sinh(\frac{3}{2}\beta z) - \lambda\sinh(\frac{1}{2}\beta z)}{\cosh(2\beta z)}\sin(xz)dz &= -\sqrt{\alpha}\frac{\sinh(\frac{3}{2}\alpha x) - \lambda\sinh(\frac{1}{2}\alpha x)}{\cosh(2\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\sinh(\frac{3}{2}\beta z) - \kappa\sinh(\frac{1}{2}\beta z)}{\cosh(2\beta z)}\sin(xz)dz &= \sqrt{\alpha}\frac{\sinh(\frac{3}{2}\alpha x) - \kappa\sinh(\frac{1}{2}\alpha x)}{\cosh(2\alpha x)} \qquad (47)
\end{aligned}$$

Here,
$$\begin{aligned}
\sigma &= 1-\sqrt{2}-\sqrt{4-2\sqrt{2}} \quad & \varepsilon &= 1-\sqrt{2}+\sqrt{4-2\sqrt{2}} \\
\lambda &= 1+\sqrt{2}+\sqrt{4+2\sqrt{2}} \quad & \kappa &= 1+\sqrt{2}-\sqrt{4+2\sqrt{2}}
\end{aligned}$$

Proposition 19. *If* $\alpha\beta = \frac{2\pi}{5}$, *then*

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(2\beta z) - \cosh(\beta z) + \phi}{\cosh(\frac{5}{2}\beta z)}\cos(xz)dz &= -\sqrt{\alpha}\frac{\cosh(2\alpha x) - \cosh(\alpha x) + \phi}{\cosh(\frac{5}{2}\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(2\beta z) + (\frac{1+\sqrt{5}}{4})}{\cosh(\frac{5}{2}\beta z)}\cos(xz)dz &= \sqrt{\alpha}\frac{\cosh(2\alpha x) + (\frac{1+\sqrt{5}}{4})}{\cosh(\frac{5}{2}\alpha x)}
\end{aligned}$$

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\cosh(\beta z)-(\frac{1+\sqrt{5}}{4})}{\cosh(\frac{5}{2}\beta z)}\cos(xz)dz &= \sqrt{\alpha}\frac{\cosh(\alpha x)-(\frac{1+\sqrt{5}}{4})}{\cosh(\frac{5}{2}\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\sinh(2\beta z)-\rho\sinh(\beta z)}{\cosh(\frac{5}{2}\beta z)}\sin(xz)dz &= -\sqrt{\alpha}\frac{\sinh(2\alpha x)-\rho\sinh(\alpha x)}{\cosh(\frac{5}{2}\alpha x)} \\
\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\beta}\frac{\sinh(2\beta z)-\eta\sinh(\beta z)}{\cosh(\frac{5}{2}\beta z)}\sin(xz)dz &= \sqrt{\alpha}\frac{\sinh(2\alpha x)-\eta\sinh(\alpha x)}{\cosh(\frac{5}{2}\alpha x)}. \quad (48)
\end{aligned}$$

where

$$\begin{aligned}
\varpi &= (\frac{\sqrt{5}-1}{2}) \\
\rho &= \frac{1}{4}(1+\sqrt{5}+\sqrt{2(5+\sqrt{5})}) \\
\eta &= \frac{1}{4}(1+\sqrt{5}-\sqrt{2(5+\sqrt{5})}).
\end{aligned}$$

Now, given these self-reciprocal functions, we may apply Poisson Simmation to generate modular forms.

Proposition 20. *If $\alpha\beta = \frac{2\pi}{m}$, and if $\tau(k)$ is a function modulo m, then*

$$\sqrt{m}\sqrt{\alpha}\sum_{n=-\infty}^{\infty}\tau(n)f(\alpha n) = \sqrt{\beta}\sum_{n=-\infty}^{\infty}\hat{\tau}(n)\hat{f}(\beta n), \quad (49)$$

where

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-itx}dt,$$

and

$$\hat{\tau}(n) = \sum_{k=0}^{m-1}\tau(k)e^{\frac{2\pi ink}{m}}.$$

Proof. We write the Poisson Summation formula in the form

$$\sum_{n=-\infty}^{\infty} g(x+n) = \sum_{n=-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(t)e^{-2\pi itn}dt\right\}e^{2\pi inx}.$$

Thus, if $\tau(k)$ is any function modulo m, we have

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-1}\tau(k)g(n+\frac{k}{m}) = \sum_{n=-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} g(t)e^{-2\pi itn}dt\right\}\sum_{k=0}^{m-1}\tau(k)e^{\frac{2\pi ink}{m}}.$$

Now put $g(\frac{x}{m}) = f(\alpha x)$ to obtain

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-1}\tau(k)f(\alpha(mn+k)) = \sum_{n=-\infty}^{\infty}\hat{\tau}(n)\left\{\int_{-\infty}^{\infty} f(\alpha mt)e^{-2\pi itn}dt\right\}.$$

Hence, since $\tau(k)$ has period m,

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{m-1}\tau(mn+k)f(\alpha(mn+k)) = \frac{\sqrt{2\pi}}{\alpha m}\sum_{n=-\infty}^{\infty}\hat{\tau}(n)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-\frac{2\pi}{\alpha m}itn}dt\right\}.$$

The proposition follows trivially. ■

Set $m=1$, in proposition 20 to obtain

$$\sqrt{\beta}\left(\frac{1}{2}F_c(0)+\sum_{n=1}^{\infty}F_c(n\beta)\right) = \sqrt{\alpha}\left(\frac{1}{2}f(0)+\sum_{n=1}^{\infty}f(n\alpha)\right), \tag{50}$$

where

$$F_c(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(z)\cos(xz)dz,$$

and $\alpha\beta = 2\pi$. Or, put $m=4$, and let τ be the nonprinciple, real, primitive Dirichlet character modulo 4. Hence, if $\alpha\beta=\frac{\pi}{2}$ then

$$\sqrt{\beta}\sum_{n=0}^{\infty}(-1)^nF_s((2n+1)\beta) = \sqrt{\alpha}\sum_{n=0}^{\infty}(-1)^nf((2n+1)\alpha), \tag{51}$$

where

$$F_s(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}f(z)\sin(xz)dz.$$

See [7] for more information. The following proposition follows easily from our previous remarks, (46), (48), and (50):

Proposition 21. *If* $\alpha\beta=\frac{4\pi^2}{3}$, *and*

$$\vartheta(\alpha) = \sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\cosh(\alpha n)-\left(\frac{1+\sqrt{3}}{2}\right)}{\cosh(\frac{3}{2}\alpha n)},$$

then

$$\vartheta(\alpha)+\vartheta(\beta) = \frac{2-\sqrt{3}}{\sqrt{3}-1}\frac{\sqrt{\alpha}+\sqrt{\beta}}{2}. \tag{52}$$

If $\alpha\beta=\frac{\pi^2}{5}$, *and*

$$\varpi(\alpha) = \sqrt{\alpha}\sum_{n=0}^{\infty}(-1)^n\frac{\sinh((4n+2)\alpha)-\rho\sinh((2n+1)\alpha)}{\cosh(5(n+\frac{1}{2})\alpha)},$$

where ρ *is as before, then*

$$\varpi(\alpha)+\varpi(\beta)=0. \tag{53}$$

As a beautiful series evaluation, set $\alpha=\beta=\frac{\pi}{\sqrt{5}}$ in (53) to obtain

$$\sum_{n=0}^{\infty}(-1)^n\frac{\sinh((2n+1)\frac{2\pi}{\sqrt{5}})}{\cosh(5(n+\frac{1}{2})\frac{\pi}{\sqrt{5}})} = \frac{1}{4}(1+\sqrt{5}+\sqrt{2(5+\sqrt{5})})\sum_{n=0}^{\infty}(-1)^n\frac{\sinh((2n+1)\frac{\pi}{\sqrt{5}})}{\cosh(5(n+\frac{1}{2})\frac{\pi}{\sqrt{5}})}. \tag{54}$$

The result

$$\sum_{n=-\infty}^{\infty} \chi(n) e^{-\frac{\pi n^2}{m} t} e^{-2\pi n z} = \frac{G(1,\chi)}{\sqrt{m}} \sqrt{\frac{1}{t}} e^{\frac{\pi m z^2}{t}} \sum_{n=-\infty}^{\infty} \bar{\chi}(n) e^{-\frac{\pi n^2}{m}\frac{1}{t}} e^{-2\pi n \frac{z}{it}}, \qquad (55)$$

where χ is a primitive dirichlet character modulo m, is well known, and is easily got from these methods. It is clear that if χ is totally real, then the function

$$\varphi(t) = \left(\sum_{n=-\infty}^{\infty} \chi(n) e^{-\frac{\pi n^2}{m} t} \right)^2$$

satisfies

$$\begin{aligned} \varphi(t+2im) &= \varphi(t), \\ \left(\frac{G(1,\chi)}{\sqrt{m}}\right)^2 \frac{1}{t}\varphi(\frac{1}{t}) &= \varphi(t). \end{aligned}$$

Also, if $f(n)$ satisfies

$$f(k) = \xi\sqrt{m} \sum_{n=1}^{m} f(n) e^{\frac{(2n-1)(2k-1)}{2m}\pi i},$$

where ξ has modulus 1, and if χ is a real Dirichlet character modulo m, then by (43) and (49),

$$\frac{\bar{\xi} i\sqrt{m}}{G(1,\chi)} \sum_{n=-\infty}^{\infty} \chi(n) \frac{\sum_{k=1}^{m} f(k) e^{(2k-1)\frac{\pi n}{m} t}}{e^{2\pi n t}+1} = \frac{1}{t} \sum_{n=-\infty}^{\infty} \chi(k) \frac{\sum_{n=1}^{m} f(k) e^{(2n-1)\frac{\pi n}{m}\frac{1}{t}}}{e^{2\pi n \frac{1}{t}}+1},$$

so that

$$\psi(t) = \sum_{n=-\infty}^{\infty} \chi(n) \frac{\sum_{k=1}^{m} f(k) e^{(2k-1)\frac{\pi n}{m} t}}{e^{2\pi n t}-1}$$

satisfies

$$\begin{aligned} \psi(t+2im) &= \psi(t), \\ \frac{G(1,\chi)}{\bar{\xi} i\sqrt{m}} \frac{1}{t}\psi(\frac{1}{t}) &= \psi(t). \end{aligned}$$

It would be interesting to find a basis for the space of all functions satisfying the above condition. In the particular case that $m = 1$, the resulting space is one dimensional. This leads directly to a proof of Jacobi's two square theorem.

We next consider a generalization of some integrals of Ramanujan. [6] The methods below are presented by Titchmarsh for the case $m = 4$.

Theorem 22. *If $\alpha\beta = \frac{2\pi}{m}$ and $\chi(n)$ is a real, primitive Dirichlet Character modulo m, then*

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\beta} e^{-\frac{i}{2}\frac{\beta}{\alpha}t^2} \frac{\sum_{n=1}^{m-1} \chi(n) e^{\beta n t}}{e^{\beta m t}-1} e^{-ixt} dt \\ = &\sqrt{\alpha} \frac{\sum_{n=1}^{m-1} \chi(n) e^{\alpha n x} \left\{ \frac{G(1,\chi)}{i\sqrt{m}} e^{\frac{i\pi n^2}{m}} - \chi(-1) e^{-\frac{i\pi}{4} - \frac{i\pi n^2}{m} + \frac{i}{2}\frac{\alpha}{\beta}x^2} \right\}}{(-1)^m e^{\alpha m x}-1}. \end{aligned} \qquad (56)$$

Proof. It is well known that if

$$\begin{aligned} F(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iut}dt \\ G(u) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-iut}dt, \end{aligned}$$

then [5]

$$\int_{-\infty}^{\infty} f(t)g(t)e^{-ixt}dt = \int_{-\infty}^{\infty} G(u)F(x-u)du. \tag{57}$$

Now if

$$f(t) = e^{-i\pi yt^2},$$

then

$$F(u) = \frac{e^{-\frac{i\pi}{4}+\frac{iu^2}{4\pi y}}}{\sqrt{2\pi y}}.$$

and by (39), if $\alpha\beta = \frac{2\pi}{m}$ and χ is a totally real primitive Dirichlet character modulo m then if

$$g(t) = i\sqrt{m}\sqrt{\beta}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\beta nt}}{e^{\beta mt}-1},$$

then

$$G(u) = G(1,\chi)\sqrt{\alpha}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\alpha nu}}{e^{\alpha mu}-1}.$$

Hence, by (57),

$$\int_{-\infty}^{\infty} e^{-i\pi yt^2}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\beta nt}}{e^{\beta mt}-1}e^{-ixt}dt = \frac{G(1,\chi)e^{-\frac{i\pi}{4}}}{i\sqrt{m}\sqrt{\beta}}\sqrt{\frac{\alpha}{2\pi y}}\int_{-\infty}^{\infty} e^{\frac{i(x-u)^2}{4\pi y}}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\alpha nu}}{e^{\alpha mu}-1}du.$$

so that if $\alpha\beta = \frac{2\pi}{m}$, then

$$\int_{-\infty}^{\infty} e^{-\frac{i}{2}\frac{\beta}{\alpha}t^2}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\beta nt}}{e^{\beta mt}-1}e^{-ixt}dt = \frac{G(1,\chi)e^{-\frac{i\pi}{4}+\frac{i}{2}\frac{\alpha}{\beta}x^2}}{i\sqrt{m}}\int_{-\infty}^{\infty} e^{\frac{i}{2}\frac{\beta}{\alpha}u^2}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\beta nu}}{e^{\beta mu}-1}e^{-ixu}du. \tag{58}$$

Thus, if

$$\phi(x) = \int_{-\infty}^{\infty} e^{-\frac{i}{2}\frac{\beta}{\alpha}t^2}\frac{\sum_{n=1}^{m-1}\chi(n)e^{\beta nt}}{e^{\beta mt}-1}e^{-ixt}dt,$$

then

$$\begin{aligned} \phi(x+mi\beta)-\phi(x) &= \sum_{n=1}^{m-1}\chi(n)\int_{-\infty}^{\infty} e^{-\frac{i}{2}\frac{\beta}{\alpha}t^2+(\beta n-ix)t}dt \\ &= \sum_{n=1}^{m-1}\chi(n)\int_{-\infty}^{\infty} e^{-\frac{i}{2}\frac{\beta}{\alpha}(t-\frac{\alpha(\beta n-ix)}{i\beta})^2-\frac{i\alpha(\beta n-ix)^2}{2\beta}}dt \\ &= \sqrt{\frac{2\pi\alpha}{\beta}}e^{-\frac{i\pi}{4}+\frac{i}{2}\frac{\alpha}{\beta}x^2}\sum_{n=1}^{m-1}\chi(n)e^{-\frac{i\pi n^2}{m}-\alpha nx}, \end{aligned} \tag{59}$$

and by (58),

$$\begin{aligned}
&\frac{i\sqrt{m}}{G(1,\chi)}e^{\frac{i\pi}{4}}\left\{e^{-\frac{i}{2}\frac{\alpha}{\beta}(x+mi\beta)^2}\phi(x+mi\beta)-e^{-\frac{i}{2}\frac{\alpha}{\beta}x^2}\phi(x)\right\}\\
=&\sum_{n=1}^{m-1}\chi(n)\int_{-\infty}^{\infty}e^{\frac{i}{2}\frac{\beta}{\alpha}u^2+(\beta n-ix)u}du\\
=&\sum_{n=1}^{m-1}\chi(n)\int_{-\infty}^{\infty}e^{\frac{i}{2}\frac{\beta}{\alpha}(u+\frac{\alpha(\beta n-ix)}{i\beta})^2+\frac{i\alpha(\beta n-ix)^2}{2\beta}}\\
=&\sqrt{\frac{2\pi\alpha}{\beta}}e^{\frac{i\pi}{4}-\frac{i}{2}\frac{\alpha}{\beta}x^2}\sum_{n=1}^{m-1}\chi(n)e^{\frac{i\pi n^2}{m}+\alpha nx}.
\end{aligned}$$

Hence,

$$\frac{i\sqrt{m}}{G(1,\chi)}\{(-1)^m e^{\alpha mx}\phi(x+mi\beta)-\phi(x)\}=\sqrt{\frac{2\pi\alpha}{\beta}}\sum_{n=1}^{m-1}\chi(n)e^{\frac{i\pi n^2}{m}+\alpha nx}. \tag{60}$$

Eliminating $\phi(x+mi\beta)$ from (59) and (60) gives

$$((-1)^m e^{\alpha mx}-1)\,\phi(x)=\sqrt{\frac{2\pi\alpha}{\beta}}\sum_{n=1}^{m-1}\chi(n)e^{\alpha nx}\left\{\frac{G(1,\chi)}{i\sqrt{m}}e^{\frac{i\pi n^2}{m}}-\chi(-1)e^{-\frac{i\pi}{4}-\frac{i\pi n^2}{m}+\frac{i}{2}\frac{\alpha}{\beta}x^2}\right\},$$

from which the theorem follows. ■

Clearly, an analogous result may be got by employing (43). By combining Theorem 22 with our previous results, we deduce the following examples:

Proposition 23. *If $\alpha\beta=\frac{\pi}{2}$ then*

$$\int_0^\infty\sqrt{\beta}\frac{\cos(\frac{\beta}{\alpha}\frac{u^2}{2})+\sin(\frac{\beta}{\alpha}\frac{u^2}{2})}{\cosh(\beta u)}\cos(xu)du=\sqrt{\frac{\pi}{2}}\sqrt{\alpha}\frac{\cos(\frac{\alpha}{\beta}\frac{x^2}{2})+\sin(\frac{\alpha}{\beta}\frac{x^2}{2})}{\cosh(\alpha x)}. \tag{61}$$

If $\alpha\beta=\frac{\pi}{3}$ then

$$\begin{aligned}
&\int_0^\infty\sqrt{\beta}\frac{\sqrt{2+\sqrt{2-\sqrt{3}}}\cos(\frac{\beta}{\alpha}\frac{u^2}{2})+\sqrt{2-\sqrt{2-\sqrt{3}}}\sin(\frac{\beta}{\alpha}\frac{u^2}{2})+\sqrt{2+\sqrt{2}}}{2\cosh(\beta u)-\frac{1}{2}\operatorname{sech}(\beta u)}\cos(xu)du\\
=&\sqrt{\frac{\pi}{2}}\sqrt{\alpha}\frac{\sqrt{2+\sqrt{2-\sqrt{3}}}\cos(\frac{\alpha}{\beta}\frac{x^2}{2})+\sqrt{2-\sqrt{2-\sqrt{3}}}\sin(\frac{\alpha}{\beta}\frac{x^2}{2})+\sqrt{2+\sqrt{2}}}{2\cosh(\alpha x)-\frac{1}{2}\operatorname{sech}(\alpha x)}.
\end{aligned} \tag{62}$$

If $\alpha\beta=\frac{\pi}{6}$ then

$$\begin{aligned}
&\int_0^\infty\sqrt{\beta}\left(\sqrt{3}\cos(\frac{\beta}{\alpha}\frac{u^2}{2})+\sin(\frac{\beta}{\alpha}\frac{u^2}{2})+\sqrt{2+\sqrt{3}}\right)\frac{\sinh(5\beta u)+\sinh(\beta u)}{\sinh(6\beta u)}\cos(xu)du\\
=&\sqrt{\frac{\pi}{2}}\sqrt{\alpha}\left(\sqrt{3}\cos(\frac{\alpha}{\beta}\frac{x^2}{2})+\sin(\frac{\alpha}{\beta}\frac{x^2}{2})+\sqrt{2+\sqrt{3}}\right)\frac{\sinh(5\alpha x)+\sinh(\alpha x)}{\sinh(6\alpha x)},
\end{aligned} \tag{63}$$

$$\int_0^\infty \left((\sqrt{3}+1)\cos(\frac{\beta}{\alpha}\frac{u^2}{2}) - (\sqrt{3}-1)\sin(\frac{\beta}{\alpha}\frac{u^2}{2}) - \sqrt{2}\right) \frac{\cosh(5\beta u) - \cosh(\beta u)}{\sinh(6\beta u)} \sin(xu)du$$
$$= -\sqrt{\frac{\pi}{2}}\sqrt{\frac{\alpha}{\beta}} \left((\sqrt{3}+1)\cos(\frac{\alpha}{\beta}\frac{x^2}{2}) - (\sqrt{3}-1)\sin(\frac{\alpha}{\beta}\frac{x^2}{2}) - \sqrt{2}\right) \frac{\cosh(5\alpha x) - \cosh(\alpha x)}{\sinh(6\alpha x)} \quad (64)$$

and

$$\int_0^\infty \sqrt{\beta}\left((\sqrt{3}-1)\cos(\frac{\beta}{\alpha}\frac{u^2}{2}) + (\sqrt{3}+1)\sin(\frac{\beta}{\alpha}\frac{u^2}{2})\right) \frac{\cosh(5\beta u) - \cosh(\beta u)}{\sinh(6\beta u)} \sin(xu)du$$
$$= \sqrt{\frac{\pi}{2}}\sqrt{\alpha} \left((\sqrt{3}-1)\cos(\frac{\alpha}{\beta}\frac{x^2}{2}) + (\sqrt{3}+1)\sin(\frac{\alpha}{\beta}\frac{x^2}{2})\right) \frac{\cosh(5\alpha x) - \cosh(\alpha x)}{\sinh(6\alpha x)}. \quad (65)$$

The first example above is essentially due to Ramanujan. The above functions could be used, as before, to generate some truly non-standard looking modular forms. We leave the amusing excercise of determining what these are in a standard basis to the reader. See [6] for related results.

5. EPILOGUE: ABSOLUTE INVARIANTS

I would like to end this paper by showing how our results on Fourier Transforms may be used to generalize our work of Section 1. Let χ be a primitive Dirichlet Character modulo m. From the well known results

$$\int_0^\infty e^{-at}\cos(bt)dt = \frac{a}{a^2+b^2},$$
$$\int_0^\infty e^{-at}\sin(bt)dt = \frac{b}{a^2+b^2},$$

we have by (49) that

$$\Psi_\chi(z) = \sum_{n=-\infty}^{\infty} \frac{\chi(n)}{n+z} = \frac{2\pi i\chi(-1)}{e^{2\pi i z}-1}\frac{G(z,\bar{\chi})}{G(1,\bar{\chi})},$$

where

$$G(z,\bar{\chi}) = \sum_{n=1}^{m} \bar{\chi}(n)e^{\frac{2\pi i n z}{m}}.$$

It is clear, then, that we could apply the theory of residues to

$$\varphi(z)\Psi_{\chi_1}(z)\Psi_{\chi_2}(zi\tau)$$

where τ is any real number and φ is a suitable meromorphic function to obtain transformation formulae of the form

$$\frac{2\pi i\chi_2(-1)}{G(1,\bar{\chi}_2)} \sum_{n=-\infty}^{\infty} \chi_1(n)\varphi(n)\frac{G(ni\tau,\bar{\chi}_2)}{e^{-2\pi n\tau}-1} + \frac{2\pi i\chi_1(-1)}{G(1,\bar{\chi}_1)} \sum_{n=-\infty}^{\infty} \chi_2(n)\frac{\varphi(\frac{ni}{\tau})}{i\tau}\frac{G(\frac{ni}{\tau},\bar{\chi}_1)}{e^{-\frac{2\pi n}{\tau}}-1}$$
$$= \frac{4\pi^2\chi_1(-1)\chi_2(-1)}{G(1,\bar{\chi}_1)G(1,\bar{\chi}_2)} \sum_\rho res_\rho(\varphi)\frac{G(\rho,\bar{\chi}_1)}{e^{2\pi i\rho}-1}\frac{G(it\rho,\bar{\chi}_2)}{e^{-2\pi t\rho}-1} \quad (66)$$

where the final sum is over all poles ρ of φ and $res_\rho(\varphi)$ is the residue of φ at ρ. See, for example, [3] It is obvious that all of our techniques from section 1 can be applied here, and that many general and beautiful results may be got from this. To illustrate, we present some "character" analogues to Theorem 4. First we will need a simple lemma:

Lemma 24. *If $\chi(n)$ is a primitive dirichlet character modulo m, and ζ is a primitive m^{th} root of unity, then*

$$\int \frac{\sum_{k=1}^{m-1}\chi(k)e^{\frac{2\pi knt}{m}}}{e^{2\pi nt}-1}dt = \frac{m}{2\pi nG(1,\bar{\chi})}\sum_{k=1}^{m-1}\bar{\chi}(k)\log(1-\zeta^k e^{-\frac{2\pi nt}{m}}).$$

Proof. Expand the left hand side as a geometric series and integrate termwise to get

$$\begin{aligned}\int \frac{\sum_{k=1}^{m-1}\chi(k)e^{\frac{2\pi knt}{m}}}{e^{2\pi nt}-1}dt &= -\frac{m}{2\pi n}\sum_{j=0}^{\infty}\sum_{k=1}^{m-1}\chi(k)\frac{e^{-\frac{2\pi nt}{m}(mj+k)}}{mj+k}\\ &= -\frac{m}{2\pi n}\sum_{j=1}^{\infty}\chi(j)\frac{e^{-\frac{2\pi njt}{m}}}{j}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{m-1}\bar{\chi}(k)\log(1-\zeta^j e^{-\frac{2\pi nt}{m}}) &= -\sum_{k=1}^{m-1}\bar{\chi}(k)\sum_{j=1}^{\infty}\frac{\zeta^{kj}e^{-\frac{2\pi njt}{m}}}{j}\\ &= -\sum_{j=1}^{\infty}\left(\sum_{k=1}^{m-1}\bar{\chi}(k)\zeta^{kj}\right)\frac{e^{-\frac{2\pi njt}{m}}}{j}\\ &= -G(1,\bar{\chi})\sum_{j=1}^{\infty}\chi(j)\frac{e^{-\frac{2\pi njt}{m}}}{j},\end{aligned}$$

from which the proposed equality easily follows. ■

Proposition 25. *Let*

$$f(\alpha)=\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\text{Arctanh}(\frac{\sqrt{3}}{1+2e^{n\alpha}}),$$

where $\left(\frac{n}{3}\right)$ is the Legendre Symbol modulo 3. If $\alpha\beta=\frac{4\pi^2}{9}$, then

$$f(\alpha)+f(\beta)=\frac{\pi}{18}.$$

Proof. By letting $\chi(n)=\left(\frac{n}{3}\right)$ in (66), we are led to examine the function

$$\varphi(z)=\pi z\frac{\sin(\frac{\pi z}{3})\sinh(\frac{\pi zt}{3})}{\sin(\pi z)\sinh(\pi zt)},$$

where t is real and positive. We have

$$\begin{aligned}R_n(\varphi) &= n(-1)^n\sin(\frac{\pi n}{3})\frac{\sinh(\frac{\pi nt}{3})}{\sinh(\pi nt)}=\frac{\sqrt{3}}{2}\left(\frac{n}{3}\right)\frac{n\sinh(\frac{\pi nt}{3})}{\sinh(\pi nt)},\\ R_{\frac{ni}{t}}(\varphi) &= -\frac{n}{t^2}(-1)^n\sin(\frac{\pi n}{3})\frac{\sinh(\frac{\pi n}{3t})}{\sinh(\frac{\pi n}{t})}=-\frac{\sqrt{3}}{2}\left(\frac{n}{3}\right)\frac{n\sinh(\frac{\pi n}{3t})}{t^2\sinh(\frac{\pi n}{t})},\end{aligned}$$

for each nonzero integer n. Hence, by the theory of Residues,

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{n\sinh(\frac{\pi nt}{3})}{\sinh(\pi nt)}-\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{n\sinh(\frac{\pi n}{3t})}{t^2\sinh(\frac{\pi n}{t})}=0. \tag{67}$$

An application of our lemma gives

$$\int\frac{n\sinh(\frac{\pi nt}{3})}{\sinh(\pi nt)}dt=\frac{\sqrt{3}}{\pi}\operatorname{Arctan}(\frac{\tanh(\frac{n\pi t}{3})}{\sqrt{3}}).$$

Thus, we have

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\left(\operatorname{Arctan}\left(\frac{\tanh(\frac{n\pi t}{3})}{\sqrt{3}}\right)+a_n\right)+\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\left(\operatorname{Arctan}\left(\frac{\tanh(\frac{n\pi}{3t})}{\sqrt{3}}\right)+b_n\right)=C.$$

After some work, one sees that $a_n=b_n=-\frac{\pi}{6}=-\operatorname{Arctan}(\frac{1}{\sqrt{3}})$, and that $C=\frac{\pi}{18}$. By using the elementary formula

$$\operatorname{Arctan}(u)-\operatorname{Arctan}(v)=\operatorname{Arctan}(\frac{u-v}{1+uv}),$$

and the identity

$$\frac{\frac{\tanh(\frac{n\pi t}{3})}{\sqrt{3}}-\frac{1}{\sqrt{3}}}{1+\frac{\tanh(\frac{n\pi t}{3})}{3}}=\frac{\sqrt{3}}{1+2e^{\frac{2n\pi t}{3}}},$$

we complete the proof. ■

Proposition 26. *Let*

$$f(\alpha)=\prod_{n=1}^{\infty}\left(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh(n\alpha)}\right)^{(\frac{n}{5})}.$$

If $\alpha\beta=\frac{4\pi^2}{25}$, *then*

$$f(\alpha)=f(\beta).$$

Proof. By letting $\chi(n)=(\frac{n}{5})$ in (60), we are led to examine the function

$$\varphi(z)=\pi z\frac{(\cos(\frac{3\pi z}{5})-\cos(\frac{\pi z}{5}))(\cosh(\frac{3\pi zt}{5})-\cosh(\frac{\pi zt}{5}))}{\sin(\pi z)\sinh(\pi zt)}$$

where t is real and positive. We have

$$\begin{aligned}R_n(\varphi) &= n(-1)^n(\cos(\frac{3\pi n}{5})-\cos(\frac{\pi n}{5}))\frac{\cosh(\frac{3\pi nt}{5})-\cosh(\frac{\pi nt}{5})}{\sinh(\pi nt)}\\ &= \frac{\sqrt{5}}{2}\left(\frac{n}{5}\right)n\frac{\cosh(\frac{3\pi nt}{5})-\cosh(\frac{\pi nt}{5})}{\sinh(\pi nt)},\\ R_{\frac{ni}{t}}(\varphi) &= \frac{n}{t^2}(-1)^n(\cos(\frac{3\pi n}{5})-\cos(\frac{\pi n}{5}))\frac{\cosh(\frac{3\pi n}{5t})-\cosh(\frac{\pi n}{5t})}{\sinh(\frac{\pi n}{t})}\\ &= \frac{\sqrt{5}}{2}\left(\frac{n}{5}\right)\frac{n}{t^2}\frac{\cosh(\frac{3\pi n}{5t})-\cosh(\frac{\pi n}{5t})}{\sinh(\frac{\pi n}{t})},\end{aligned}$$

for each nonzero integer n. Hence, by the theory of Residues,

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)n\frac{\cosh(\frac{3\pi nt}{5})-\cosh(\frac{\pi nt}{5})}{\sinh(\pi nt)}+\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{n}{t^2}\frac{\cosh(\frac{3\pi n}{5t})-\cosh(\frac{\pi n}{5t})}{\sinh(\frac{\pi n}{t})}=0. \qquad (68)$$

The integral

$$\int n\frac{\cosh(\frac{3\pi nt}{5})-\cosh(\frac{\pi nt}{5})}{\sinh(\pi nt)}dt=\frac{\sqrt{5}}{2\pi}\log(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh(\frac{2\pi nt}{5})})$$

again follows easily from Lemma 24. Thus, we have

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\log(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh(\frac{2\pi nt}{5})})=\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\log(1-\frac{2\sqrt{5}}{1+\sqrt{5}+4\cosh(\frac{2\pi n}{5t})})$$

from which the proposition follows easily. Notice that $f(\alpha)$ is absolutely invariant under the transformations $\alpha\to\alpha+2\pi i$, and $\alpha\to\frac{4\pi^2}{25\alpha}$. ■

Proposition 27. *Define* $\chi_{12}(n)=\left\{\begin{array}{c}1 \text{ if } n\equiv\pm1\ (12)\\ -1 \text{ if } n\equiv\pm5\ (12)\\ 0 \text{ otherwise}\end{array}\right\}$, *and let*

$$f(\alpha)=\prod_{n=1}^{\infty}\left(1+\frac{2\sqrt{3}}{2\cosh(n\alpha)-\sqrt{3}}\right)^{\chi_{12}(n)}.$$

If $\alpha\beta=\frac{\pi^2}{36}$, *then*

$$f(\alpha)=f(\beta).$$

Proof. The proof proceeds along the same lines as above, so we omit it. ■

Proposition 28. *Define* $\chi_{8}(n)=\left\{\begin{array}{c}1 \text{ if } n\equiv\pm1\ (8)\\ -1 \text{ if } n\equiv\pm3\ (8)\\ 0 \text{ otherwise}\end{array}\right\}$, *and let*

$$f(\alpha)=\prod_{n=1}^{\infty}\left(\frac{2\sqrt{2}}{\sqrt{2}-\operatorname{sech}(n\alpha)}-1\right)^{\chi_8(n)}.$$

If $\alpha\beta=\frac{\pi^2}{16}$, *then*

$$f(\alpha)=f(\beta).$$

Moreover,

$$f(\frac{\pi}{4})=1+\sqrt{2}.$$

Proof. Again, the proof is similar to before. The stated value of $f(\frac{\pi}{4})$ is a conjecture. ■

Proposition 29. *Let*

$$f(\alpha)=\prod_{n=1}^{\infty}\left(1-\frac{2\sqrt{17}(1+\cosh(n\alpha)+\cosh(2n\alpha)+\cosh(3n\alpha))}{2+\sqrt{17}+(7+\sqrt{17})\cosh(n\alpha)+(5+\sqrt{17})\cosh(2n\alpha)+(1+\sqrt{17})\cosh(3n\alpha)+2\cosh(4n\alpha)}\right)^{\left(\frac{n}{17}\right)}.$$

If $\alpha\beta=\frac{4\pi^2}{289}$, *then*

$$f(\alpha)=f(\beta).$$