In these notes we describe field extensions of local fields with perfect residue field, with special attention to $\mathbf{Q}_p$.

1 Unramified Extensions

Definition 1.1. An extension L/K of local fields is unramified if $[L:K] = [l:k]$ with $l = \mathcal{O}_L/\pi_L$ and $K = \mathcal{O}_K/\pi_K$ where π_L, π_K are uniformizers of L, K. This is equivalent to saying that π_K is inert in L, i.e. that the ramification index $e = v_K(\pi_L)$ is 1.

Theorem 1.2. *Fix a local field K with perfect residue field k. Then there is an equivalence of categories between the extensions of k and the unramified extensions of K.*

Proof. Consider the functor $F(L) = l := \mathcal{O}_L/\pi_L$ for any unramified extension L of K. Any K-morphism $\sigma \in \mathrm{Hom}_K(L, L')$ yields, by restriction to $\mathcal{O}_L$ a morphism $\mathcal{O}_L \to \mathcal{O}_{L'}$ preserving $\mathcal{O}_K$, so it must be a *local* map of local rings, and hence induces a field morphism $\overline{\sigma} \in \mathrm{Hom}_k(l, l')$. The map so obtained $\mathrm{Hom}_K(L, L') \to \mathrm{Hom}_k(l, l')$ obviously preserves the identity morphism.

We describe the functor inverse to F. Namely, if l/k is an extension of k, then since k is perfect, we have $l = k[\alpha]$ with α having minimum polynomial $g(x) \in k[x]$. Choose a lift $G(x)$ of g and let β be the unique root of G in $\overline{K}$ reducing to α (guaranteed by Hensel's Lemma). Define $L = K[\beta]$. Then $\mathcal{O}_L/\pi_L = l$, and since g irreducible implies G irreducible, we have $[L:K] = \deg G = \deg g = [l:k]$.

It remains to show that any k-morphism $l \to l'$ can be lifted to a K-morphism $L \to L'$. This follows from Hensel's Lemma. ∎

Example 1.3. Consider the finite unramified extensions of $\mathbf{Q}_p$. By the above theorem, these are in 1-1 correspondence with finite extensions of $\mathbf{F}_p$. But $\mathbf{F}_p$ has a unique extension of degree n for every n, namely the splitting field of $x^{p^n} - x$. It follows that $\mathbf{Q}_p$ has a *unique* unramified extension of degree n for each n, obtained as the splitting field of $x^{p^n} - x$, i.e. by adjoining the $p^n - 1$ st roots of unity. Moreover, the maximal unramified extension $\mathbf{Q}_p^{\mathrm{nr}}$ of $\mathbf{Q}_p$ corresponds to the separable (=algebraic) closure of $\mathbf{F}_p$, and so is obtained by adjoining the $p^n - 1$ st roots of unity for all n. For any integer n with $(n,p) = 1$, we have $p^{\varphi(n)} - 1 \equiv 0 \bmod n$, so we see that $\mathbf{Q}_p^{\mathrm{nr}}$ is obtained by adjoining the n th roots of unity for $(n,p) = 1$ for all n, i.e. as the direct limit of the fields $\mathbf{Q}_p(\zeta_n)$.

Remark 1.4. For any local field K the maximal unramified extension corresponds to k^{sep} (which equals $\overline{k}$ when k is perfect), and this is obtained in the case of finite k by adjoining all roots of unity prime to $\mathrm{char}\, k$.

Example 1.5. As we have just seen, $\mathbf{Q}_p^{\mathrm{nr}} = \varinjlim_{(n,p)=1} K_n$ with $K_n = \mathbf{Q}_p(\zeta_{p^n-1})$, and since K_n is unramified over $\mathbf{Q}_p$, we have an isomorphism

$$\mathrm{Gal}(K_n/\mathbf{Q}_p) \simeq \mathrm{Gal}(\mathbf{F}_{p^n}/\mathbf{F}_p) \simeq \mathbf{Z}/n\mathbf{Z}$$

so $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}/\mathbf{Q}_p) \simeq \varprojlim_n \mathbf{Z}/n\mathbf{Z} \simeq \widehat{\mathbf{Z}}$. Explicitly, for $\alpha \in \widehat{\mathbf{Z}}$ we have $\alpha(\zeta_n) = \zeta_n^a$ where $a \in \mathbf{Z}$ is "close enough" to α (that is, $a \equiv \alpha \bmod n$).

2 Totally Ramified Extensions

Definition 2.1. Let L/K be a finite extension of nonarchimedean local fields. Then L/K is totally ramified if $e = v_L(\pi_K) = [L:K]$, or equivalently if $l = k$.

Definition 2.2. A nonzero polynomial $a_0X^n + a_1X^{n-1} + \cdots + a_n \in \mathcal{O}_K[X]$ is Eisenstein if $v_K(a_0) = 0$, $v_K(a_i) \geq 1$ for $1 \leq i < n$ and $v_K(a_n) = 1$.

Proposition 2.3. *An Eisenstein polynomial is irreducible.*

Proof. Look at its factorization in $k[X]$ and conclude that $v_K(a_n) \geq 2$ by lifting; a contradiction. ∎

Theorem 2.4. *An extension L/K of nonarch local fields is totally ramified iff $L = K[\alpha]$, with α a root of an Eisenstein polynomial.*

Proof. Let f be Eisenstein of degree n. The condition on the valuations of the coefficients implies that the newton polygon of f has one line segment of slope $1/n$, and hence that every root of f has (extended) valuation $1/n$. This implies that the ramification index is at least n, and since it can be at most $n = ef$, the extension $K[X]/f(X)$ is totally ramified.

Conversely, suppose that L/K is totally ramified and pick $\alpha \in L$ with $v_L(\alpha) = 1$. Then since for any $a_i \in K$ we have $v_L(\sum_{i=0}^{n-1} a_i\alpha^i) > 0$ (because α^i has valuation i/n, so the $a_i\alpha^i$ all have different valuation) we see that $1, \alpha, \ldots, \alpha^{n-1}$ are linearly independent over K, so α satisfies

$$a_0\alpha^n + a_1\alpha^{n-1} + \ldots + a_n = 0.$$

This implies that there exist i, j with $v_L(a_i\alpha^i) = v_L(a_j\alpha^j)$ maximal for two distinct i, j. This can only happen if $f = \sum a_i X^i$ is Eisenstein. ■

Corollary 2.5. *For every integer $e \geq 1$, every nonarchimedean local field K has a totally ramified extension of degree e.*

Proof. Pick a uniformizer $\pi \in K$. Then $X^e - \pi$ is Eisenstein. ■

2.6 Ramification Groups

Definition 2.7. Let L/K be a galois extension of nonarchimedean local fields with galois group G and let $\pi \in K$ be a uniformizer. For $i \geq 0$, the i th ramification group is

$$G^i := \{\sigma \in G : |\sigma\alpha - \alpha|\} < |\pi|^i$$

Theorem 2.8. *The groups G_i are normal subgroups of G and give a filtration $G \supset G_0 \supset G_1 \supset \ldots$. Moreover, $G_i = 1$ for all suff large i and there are injections $G_0/G_1 \hookrightarrow l^\times$ and $G_i/G_{i+1} \hookrightarrow l$ for $i \geq 1$ and an isomorphism $G/G_0 \simeq \mathrm{Gal}(l/k)$.*

Proof. Normality follows from the fact that $\mathrm{Gal}(L/K)$ preserves the valuation. That $G_i = 1$ for large i follows from the observation that $\sigma \notin G_i$ as soon as there exists $\alpha \in L$ with $\sigma\alpha - \alpha \geq |\pi|^i$.

Define $G_0 \to l^\times$ by $\sigma \mapsto \sigma(\pi)/\pi$. This gives a well defined map $G_0 \to l^\times$ as $\sigma(\pi) = u\pi$ for any $\sigma \in \mathrm{Gal}(L/K)$. The kernel of this map is G_1, so $G_0/G_1 \hookrightarrow l^\times$. We also define $G_i \to l$ by $\sigma \mapsto (\sigma(\pi) - \pi)/\pi^{i+1}$, and this gives an injection $G_i/G_{i+1} \hookrightarrow l$.

Finally, the surjective natural map $G \to \mathrm{Gal}(l/k)$ has kernel G_0 so $G/G_0 \simeq \mathrm{Gal}(l/k)$, and from our characterization of unramified extensions above, we see that the maximal unramified extension K_0 of K inside L is Galois, being the fixed field of G_0. ■

Corollary 2.9. *Let L/K be a galois extension, with Galois group G and ramification groups G_i. Then*

1. *L/K is unramified if and only if $G_0 = 1$. In particular, the ramification degree of L/K is $\#G_0$ and the maximal unramified extension of K in L is the fixed field of G_0 and has galois group G/G_0.*

2. *L/K is tamely ramified if and only if $G_1 = 1$. In particular, if we write the ramification index as $e = q^n m$ with $(m, q) = 1$ and $q = \mathrm{char}\, k$, then $\#G_1 = q^n$ and this divides $\#l$. Moreover, if K_1 is the maximal tamely ramified extension of K inside L, then K_1 is the fixed field of G_1, and is hence an extension of K_0, and has Galois group G/G_1.*

3. *We thus have a tower of fields $L \supset K_1 \supset K_0 \supset K$ where K_0/K is unramified of degree $\#G/G_0$, the extension K_1/K_0 is totally tamely ramified of degree $\#G_0/G_1$ (which divides $\#l^\times$) and L/K_1 is totally wildly ramified of degree $\#G_1$ (which divided $\#l$).*

Example 2.10. Consider the extension $\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p$. It is the splitting field of $X^{p^n}-1$, and we have an isomorphism $(\mathbf{Z}/p^n\mathbf{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^n})/\mathbf{Q}_p)$ given by $a \mapsto (\zeta \mapsto \zeta^a)$. This is a totally ramified extension of degree $\varphi(p^n) = (p-1)p^{n-1}$ as $\mathrm{Nm}(1-\zeta_{p^n}) = \lim_{x\to 1}(x^{p^n}-1)/(x^{p^{n-1}}-1) = p$, so $\pi := 1-\zeta_{p^n}$ is a uniformizer. Taking inverse limits, we have the extension $\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p$. It is totally ramified with Galois group $\mathbf{Z}_p^\times$, and $a \in \mathbf{Z}_p^\times$ acts on ζ_{p^n} by $\zeta \mapsto \zeta^{a \bmod p^s}$ for any $s \geq n$.