CHAPTER

1 A Library of Functions

1.1 Functions and Change

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In January, 2007, the temperatures in Fresno, California were unusually low for the winter and much of the orange crop was lost. The daily high temperatures for January 9–18 are given in Table 1.1.

Table 1.1 *Daily High Temperature in Fresno, January 9–18, 2007*

Date (January 2007)	9	10	11	12	13	14	15	16	17	18
High temperature (°F)	32	32	39	25	23	25	24	25	28	29

Although you may not have thought of something so unpredictable as temperature as being a function, the temperature *is* a function of date, because each day gives rise to one and only one high temperature. There is no formula for temperature (otherwise we would not need the weather bureau), but nevertheless the temperature does satisfy the definition of a function: Each date, t, has a unique high temperature, H, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the temperature example, the domain is the set of dates {9, 10, 11, 12, 13, 14, 15, 16, 17, 18} and the range is the set of temperatures {23, 24, 25, 28, 29, 32, 39}. We call the function f and write $H = f(t)$. Notice that a function may have identical outputs for different inputs (January 12, 14, and 16, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

$$\text{The set of numbers } t \text{ such that } a \le t \le b \text{ is written } [a, b].$$

$$\text{The set of numbers } t \text{ such that } a < t < b \text{ is written } (a, b).$$

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily high temperatures in Fresno can be represented by the graph in Figure 1.1, as well as by Table 1.1.

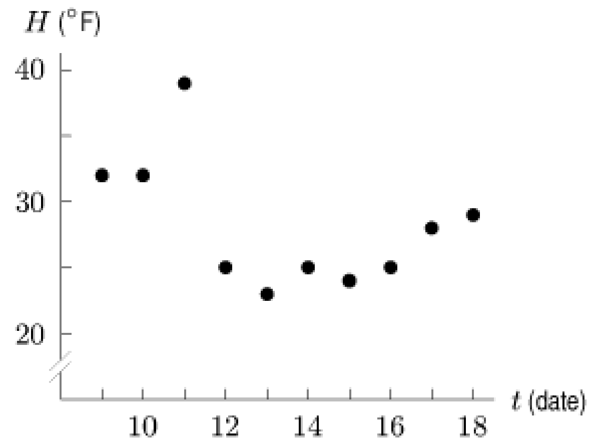


Figure 1.1 Fresno temperatures, January 2007

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed by the formula

$$C = 4T - 160$$

to a fair degree of accuracy. We write $C = f(T)$ to express the fact that we think of C as a function of T and that we have named this function f. The graph of this function is in Figure 1.2.

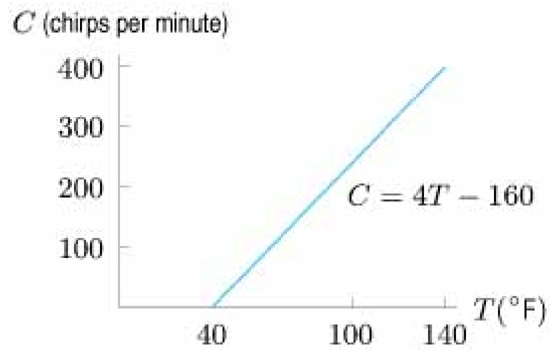


Figure 1.2 Cricket chirp rate versus temperature

Examples of Domain and Range

If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers. However, the domain of the function $g(x) = 1/x$ is all real numbers except zero, since we cannot divide by zero.

Sometimes we restrict the domain to be smaller than the largest possible set of real numbers. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to nonnegative values of x.

Example 1

The function $C = f(T)$ gives chirp rate as a function of temperature. We restrict this function to temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, 136°F. What is the domain of this function f?

Solution

If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between two variables C and T, any T value is possible. However, if we think of it as a relationship between cricket chirps and temperature, then C cannot be less than 0. Since $C = 0$ leads to $0 = 4T - 160$, and so $T = 40$°F, we see that T cannot be less than 40°F. (See Figure 1.2.) In addition, we are told that the function is not defined for temperatures above 136°. Thus, for the function $C = f(T)$ we have

$$\begin{aligned} \text{Domain} &= \text{All } T \text{ values between } 40^\circ\, F \text{ and } 136^\circ\, F \\ &= \text{All } T \text{ values with } 40 \le T \le 136 \\ &= [40, 136]. \end{aligned}$$

Example 2

Find the range of the function f, given the domain from Example 1. In other words, find all possible values of the chirp rate, C, in the equation $C = f(T)$.

Solution

Again, if we consider $C = 4T - 160$ simply as a mathematical relationship, its range is all real C values. However, when thinking of the meaning of $C = f(T)$ for crickets, we see that the function predicts cricket chirps per minute between 0 (at $T = 40°F$) and 384 (at $T = 136°F$). Hence,

$$\begin{aligned}\text{Range} &= \text{All } C \text{ values from } 0 \text{ to } 384\\ &= \text{All } C \text{ values with } 0 \le C \le 384\\ &= [0, 384].\end{aligned}$$

In using the temperature to predict the chirp rate, we thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backward, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Linear Functions

The chirp-rate function, $C = f(T)$, is an example of a *linear function*. A function is linear if its slope, or rate of change, is the same at every point. The rate of change of a function that is not linear may vary from point to point.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year. So the height was a linear function of time from 1900 to 1912. If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

Since $y = f(t)$ increases with t, we say that f is an *increasing function*. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

Table 1.2 *Men's Olympic Pole Vault Winning Height (Approximate)*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

This rate of increase is the *slope* of the line in Figure 1.3. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches / year.}$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

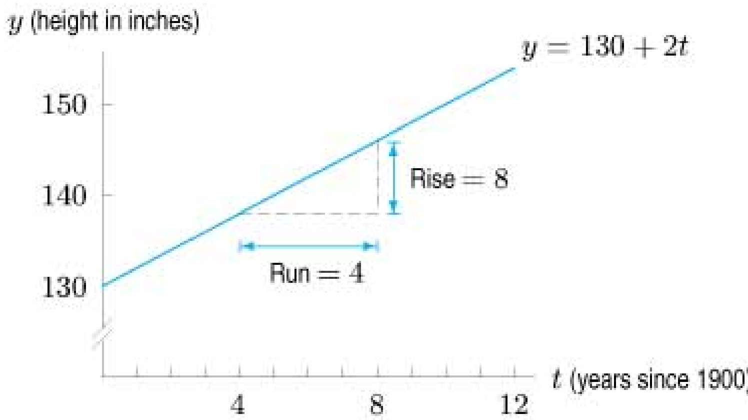


Figure 1.3 Olympic pole vault records

What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, 130 is the *intercept* on the vertical axis.

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130 + 2t$ predicts that the height in the 2004 Olympics would be 338 inches or 28 feet 2 inches, which is considerably higher than the actual value of 19 feet 6.25 inches. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 130 + 2t$ makes sense for all values of t. The graph in Figure 1.3 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

As the pole vault heights have increased over the years, the time to run the mile has decreased. If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.39t.$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.39, tells us that the world record decreased by about 0.39 seconds per year. We say that g is a *decreasing function*.

Difference Quotients and Delta Notation

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y.

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.4). Since $m = \Delta y/\Delta x$, the units of m are y-units over x-units.

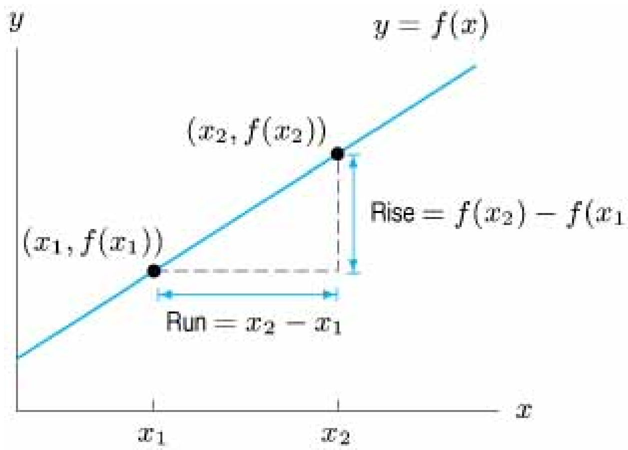


Figure 1.4 Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Families of Linear Functions

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line.

To recognize that a table of x and y values comes from a linear function, $y = b + mx$, look for differences in y-values that are constant for equally spaced x-values.

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, give a *family of functions*. All the functions in a family share certain properties—in this case, all the graphs are straight lines. The constants m and b are called *parameters*; their meaning is shown in Figures 1.5 and 1.6. Notice the greater the magnitude of m, the steeper the line.

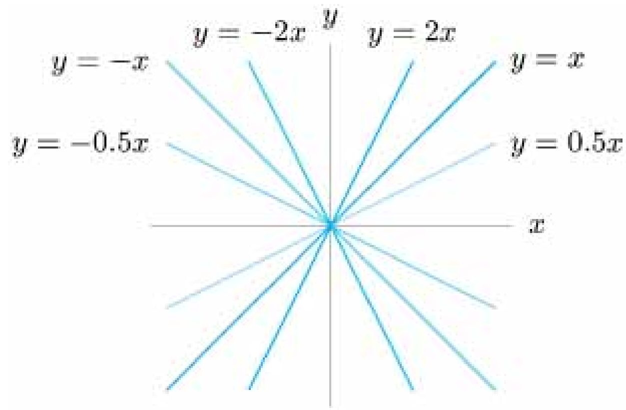


Figure 1.5 The family $y = mx$(with $b = 0$)

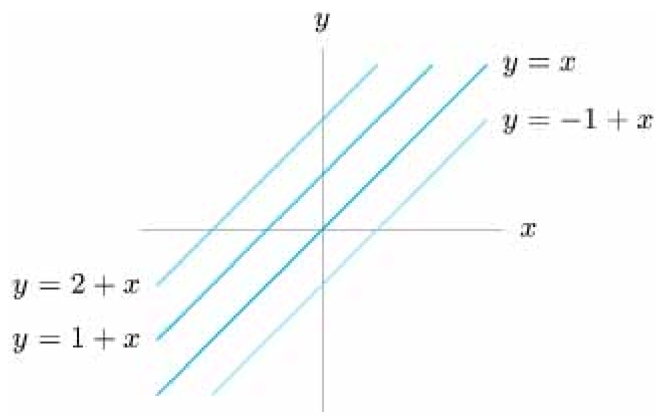


Figure 1.6 The family $y = b + x$(with $m = 1$)

Increasing Versus Decreasing Functions

The terms increasing and decreasing can be applied to other functions, not just linear ones. See Figure 1.7. In general,

A function f is **increasing** if the values of $f(x)$ increase as x increases.

A function f is **decreasing** if the values of $f(x)$ decrease as x increases.

The graph of an *increasing* function *climbs* as we move from left to right.

The graph of a *decreasing* function *falls* as we move from left to right.

A function $f(x)$ is **monotonic** if it increases for all x or decreases for all x.

Figure 1.7 Increasing and decreasing functions

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, the area, A, of a circle is proportional to the square of the radius, r, because

$$A = f(r) = \pi r^2.$$

We say y is (directly) **proportional** to x if there is a nonzero constant k such that

$$y = kx.$$

This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = 50\left(\frac{1}{t}\right) = \frac{50}{t}.$$

Exercises and Problems for Section 1.1

Exercises

1. The population of a city, P, in millions, is a function of t, the number of years since 1970, so $P = f(t)$. Explain the meaning of the statement $f(35) = 12$ in terms of the population of this city.
2. When a patient with a rapid heart rate takes a drug, the heart rate plunges dramatically and then slowly rises again as the drug wears off. Sketch the heart rate against time from the moment the drug is administered.

3. Describe what Figure 1.8 tells you about an assembly line whose productivity is represented as a function of the number of workers on the line.

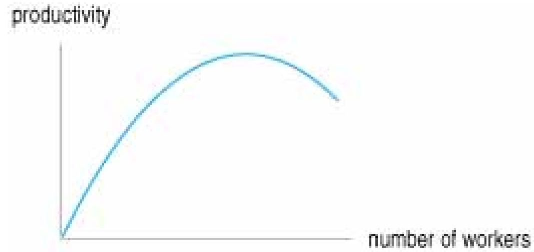


Figure 1.8

For Exercises 4, 5, 6 and 7, find an equation for the line that passes through the given points.

4. (0, 0) and (1, 1)

5. (0, 2) and (2, 3)

6. (-2, 1) and (2, 3)

7. (-1, 0) and (2, 6)

For Exercises 8, 9, 10 and 11, determine the slope and the *y*-intercept of the line whose equation is given.

8. $2y + 5x - 8 = 0$

9. $7y + 12x - 2 = 0$

10. $-4y + 2x + 8 = 0$

11. $12x = 6y + 4$

12. Match the graphs in Figure 1.9 with the following equations. (Note that the *x* and *y* scales may be unequal.)

(a) $y = x - 5$

(b) $-3x + 4 = y$

(c) $5 = y$

(d) $y = -4x - 5$

(e) $y = x + 6$

(f) $y = x/2$

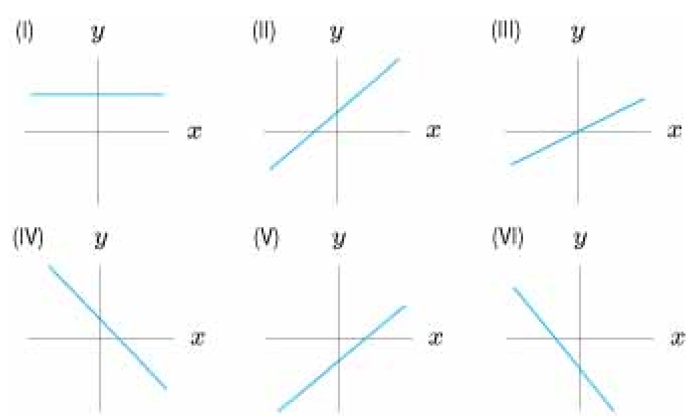


Figure 1.9

13. Match the graphs in Figure 1.10 with the following equations. (Note that the *x* and *y* scales may be unequal.)

(a) $y = -2.72x$
(b) $y = 0.01 + 0.001x$
(c) $y = 27.9 - 0.1x$
(d) $y = 0.1x - 27.9$
(e) $y = -5.7 - 200x$
(f) $y = x/3.14$

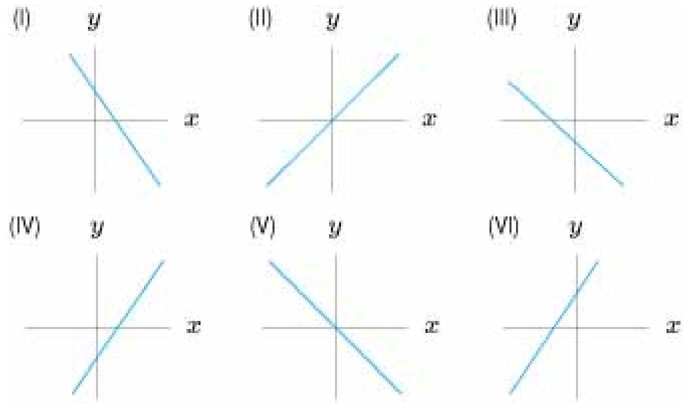


Figure 1.10

14. Estimate the slope and the equation of the line in Figure 1.11.

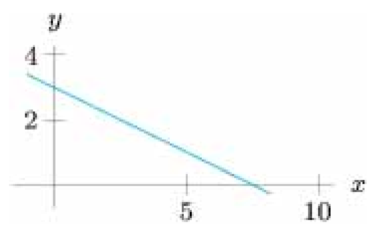


Figure 1.11

15. Find an equation for the line with slope *m* through the point (*a*, *c*).

16. Find a linear function that generates the values in Table 1.3.

Table 1.3

x	5.2	5.3	5.4	5.5	5.6
y	27.8	29.2	30.6	32.0	33.4

For Exercises 17, 18 and 19, use the facts that parallel lines have equal slopes and that the slopes of perpendicular lines are negative reciprocals of one another.

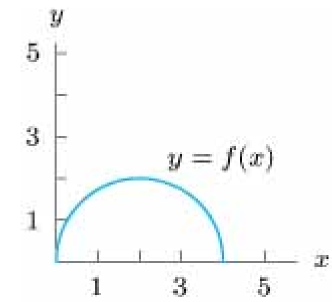
17. Find an equation for the line through the point (2, 1) which is perpendicular to the line $y = 5x - 3$.

18. Find equations for the lines through the point (1, 5) that are parallel to and perpendicular to the line with equation $y + 4x = 7$.

19. Find equations for the lines through the point (*a*, *b*) that are parallel and perpendicular to the line $y = mx + c$, assuming $m \neq 0$.

For Exercises 20, 21, 22 and 23, give the approximate domain and range of each function. Assume the entire graph is shown.

20.



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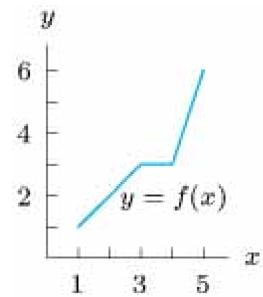

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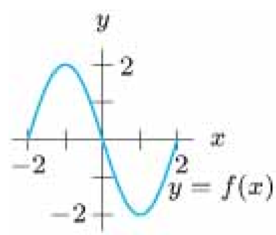

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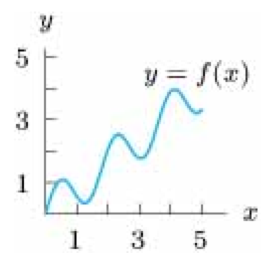

Find domain and range in Exercises 24 and 25.

24. $y = x^2 + 2$

25. $y = \dfrac{1}{x^2 + 2}$

26. If $f(t) = \sqrt{t^2 - 16}$, find all values of t for which $f(t)$ is a real number. Solve $f(t) = 3$.

27. If $g(x) = (4 - x^2)/(x^2 + x)$, find the domain of $g(x)$. Solve $g(x) = 0$.

In Exercises 28, 29, 30, 31 and 32, write a formula representing the function.

28. The volume of a sphere is proportional to the cube of its radius, *r*.

29. The average velocity, *v*, for a trip over a fixed distance, *d*, is inversely proportional to the time of travel, *t*.

30. The strength, *S*, of a beam is proportional to the square of its thickness, *h*.

31. The energy, *E*, expended by a swimming dolphin is proportional to the cube of the speed, *v*, of the dolphin.

32. The number of animal species, *N*, of a certain body length, *l*, is inversely proportional to the square of *l*.

Problems

33. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, *a*, in years.

(a) Interpret the statement $f(5) = 6$

(b) Sketch a possible graph of *V* against *a*. Is *f* an increasing or decreasing function? Explain.

(c) Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.

34. Which graph in Figure 1.12 best matches each of the following stories?[1] Write a story for the remaining graph.

(a) I had just left home when I realized I had forgotten my books, and so I went back to pick them up.

(b) Things went fine until I had a flat tire.

(c) I started out calmly but sped up when I realized I was going to be late.

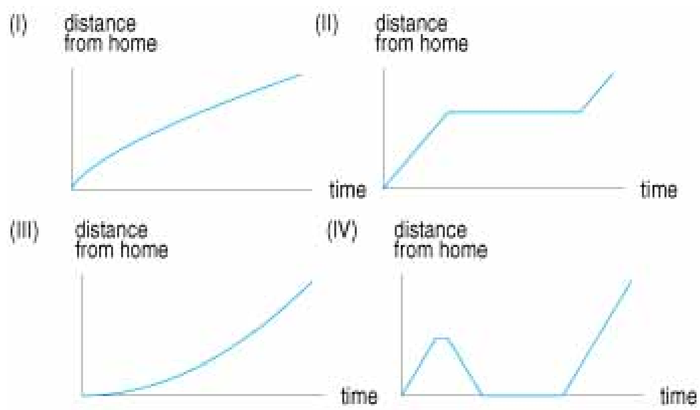

Figure 1.12

35. A company rents cars at $40 a day and 15 cents a mile. Its competitor's cars are $50 a day and 10 cents a mile.

(a) For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.

(b) On the same axes, graph both functions.

(c) How should you decide which company is cheaper?

36. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount monthly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed \$40, while one using 1600 cubic feet was billed \$55.

(a) What is the charge per cubic foot?

(b) Write an equation for the total cost of a resident's water as a function of cubic feet of water used.

(c) How many cubic feet of water used would lead to a bill of \$100?

37. An object is put outside on a cold day at time $t = 0$. Its temperature, $H = f(t)$, in °C, is graphed in Figure 1.13.

(a) What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.

(b) Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

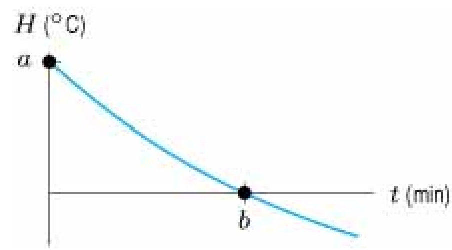

Figure 1.13

38. The force, F , between two atoms depends on the distance r separating them. See Figure 1.14. A positive F represents a repulsive force; a negative F represents an attractive force.

(a) What happens to the force if the atoms start with $r = a$ and are

(i) Pulled slightly further apart?

(ii) Pushed slightly closer together?

(b) The atoms are said to be in *stable equilibrium* if the force between them is zero and the atoms tend to return to the equilibrium after a minor disturbance. Does $r = a$ represent a stable equilibrium? Explain.

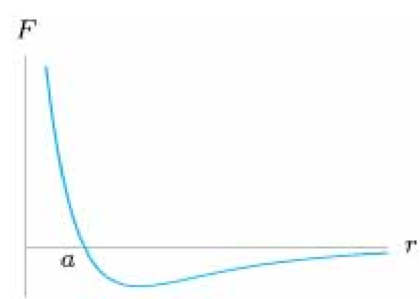

Figure 1.14

39. A controversial 1992 Danish study[2] reported that men's average sperm count has decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.

(a) Express the average sperm count, S, as a linear function of the number of years, t, since 1940.

(b) A man's fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?

40. The graph of Fahrenheit temperature, °F, as a function of Celsius temperature, °C, is a line. You know that 212°F and 100°C both represent the temperature at which water boils. Similarly, 32°F and 0°C both represent water's freezing point.

(a) What is the slope of the graph?

(b) What is the equation of the line?

(c) Use the equation to find what Fahrenheit temperature corresponds to 20°C.

(d) What temperature is the same number of degrees in both Celsius and Fahrenheit?

41. The demand function for a certain product, $q = D(p)$, is linear, where p is the price per item in dollars and q is the quantity demanded. If p increases by $5, market research shows that q drops by two items. In addition, 100 items are purchased if the price is $550.

(a) Find a formula for

(i) q as a linear function of p

(ii) p as a linear function of q

(b) Draw a graph with q on the horizontal axis.

42. The cost of planting seed is usually a function of the number of acres sown. The cost of the equipment is a *fixed cost* because it must be paid regardless of the number of acres planted. The cost of supplies and labor varies with the number of acres planted and are called *variable costs.* Suppose the fixed costs are $10,000 and the variable costs are $200 per acre. Let C be the total cost, measured in thousands of dollars, and let x be the number of acres planted.

(a) Find a formula for C as a function of x.

(b) Graph C against x.

(c) Which feature of the graph represents the fixed costs? Which represents the variable costs?

43. You drive at a constant speed from Chicago to Detroit, a distance of 275 miles. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.

44. A flight from Dulles Airport in Washington, DC, to LaGuardia Airport in New York City has to circle LaGuardia several times before being allowed to land. Plot a graph of the distance of the plane from Washington, DC, against time, from the moment of takeoff until landing.

45. (a) Consider the functions graphed in Figure 1.15(a). Find the coordinates of *C*.

(b) Consider the functions in Figure 1.15(b). Find the coordinates of *C* in terms of *b*.

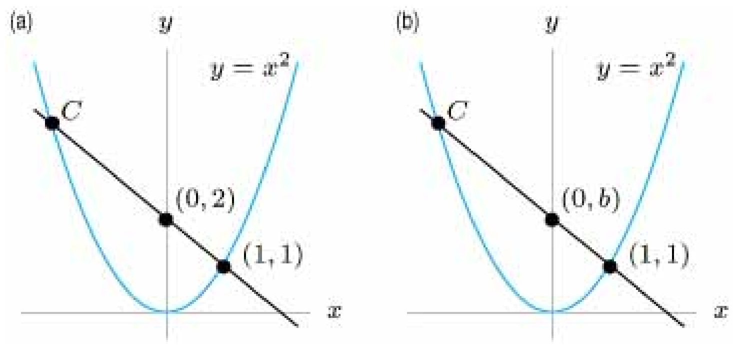

Figure 1.15

46. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen. What do the experimental data in Table 1.4 tell you about Galileo's hypothesis? What alternative hypothesis is suggested by the two sets of data in Table 1.4 and Table 1.5?

Table 1.4

Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table 1.5

Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

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