1.2 Exponential Functions

Population Growth

The population of Nevada[3] from 2000 to 2006 is given in Table 1.6. To see how the population is growing, we look at the increase in population in the third column. If the population had been growing linearly, all the numbers in the third column would be the same.

Table 1.6 *Population of Nevada (Estimated), 2000–2006*

Year	Population (millions)	Change in population (millions)
2000	2.020	
2001	2.093	0.073
2002	2.168	0.075
2003	2.246	0.078
2004	2.327	0.081
2005	2.411	0.084

2006	2.498	0.087

Suppose we divide each year's population by the previous year's population. For example,

$$\frac{\text{Population in 2001}}{\text{Population in 2000}} = \frac{2.093 \text{ million}}{2.020 \text{ million}} = 1.036$$

$$\frac{\text{Population in 2002}}{\text{Population in 2001}} = \frac{2.168 \text{ million}}{2.093 \text{ million}} = 1.036.$$

The fact that both calculations give 1.036 shows the population grew by about 3.6% between 2000 and 2001 *and* between 2001 and 2002. Similar calculations for other years show that the population grew by a factor of about 1.036, or 3.6%, every year. Whenever we have a constant growth factor (here 1.036), we have exponential growth. The population t years after 2000 is given by the *exponential* function

$$P = 2.020(1.036)^t.$$

If we assume that the formula holds for 50 years, the population graph has the shape shown in Figure 1.16. Since the population is growing faster and faster as time goes on, the graph is bending upward; we say it is *concave up*. Even exponential functions which climb slowly at first, such as this one, eventually climb extremely quickly.

To recognize that a table of t and P values comes from an exponential function $P = P_0 a^t$, look for ratios of P values that are constant for equally spaced t values.

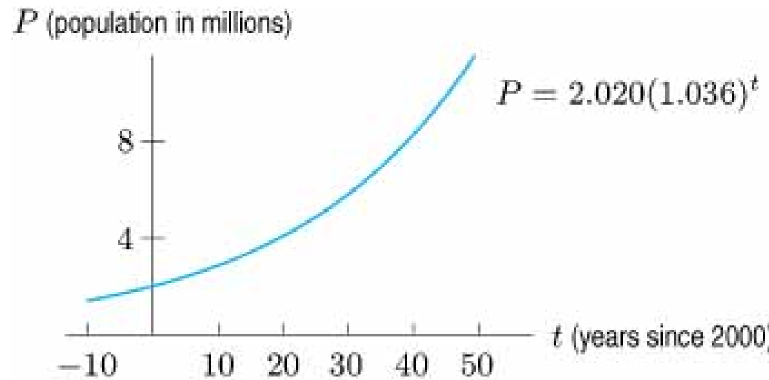


Figure 1.16 Population of Nevada (estimated): Exponential growth

Concavity

We have used the term concave up[4] to describe the graph in Figure 1.16. In words:

> The graph of a function is **concave up** if it bends upward as we move left to right; it is **concave down** if it bends downward. (See Figure 1.17 for four possible shapes.) A line is neither concave up nor concave down.

Figure 1.17 Concavity of a graph

Elimination of a Drug from the Body

Now we look at a quantity which is decreasing exponentially instead of increasing. When a patient is given medication, the drug enters the bloodstream. As the drug passes through the liver and kidneys, it is metabolized and eliminated at a rate that depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since every hour the amount remaining is 60% of the previous amount, we have

$$f(0) = 250$$
$$f(1) = 250(0.6)$$
$$f(2) = (250(0.6))(0.6) = 250(0.6)^2,$$

and after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This is an *exponential decay function*. Some values of the function are in Table 1.7; its graph is in Figure 1.18.

Table 1.7

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

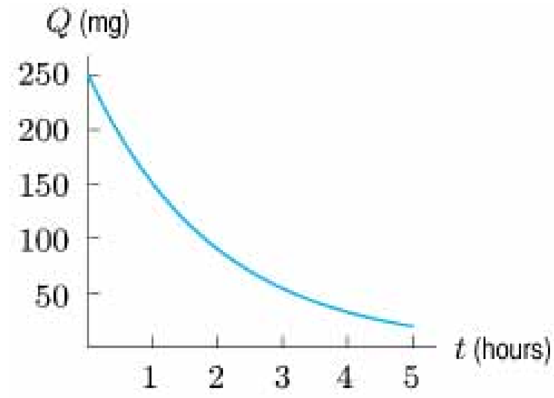


Figure 1.18 Drug elimination: Exponential decay

Notice the way the function in Figure 1.18 is decreasing. Each hour a smaller quantity of the drug is removed than in the previous hour. This is because as time passes, there is less of the drug in the body to be removed. Compare this to the exponential growth in Figure 1.16, where each step upward is larger than the previous one. Notice, however, that both graphs are concave up.

The General Exponential Function

We say P is an **exponential function** of t with base a if

$$P = P_0 a^t,$$

where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1.

If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

Provided $a > 0$, the largest possible domain for the exponential function is all real numbers. The reason we do not want $a \leq 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 1^t = P_0$ is then a constant function.

The value of a is closely related to the percent growth (or decay) rate. For example, if $a = 1.03$, then P is growing at 3%; if $a = 0.94$, then P is decaying at 6%.

Example 1

Suppose that $Q = f(t)$ is an exponential function of t. If $f(20) = 88.2$ and $f(23) = 91.4$:

(a) Find the base.

(b) Find the growth rate.

(c) Evaluate $f(25)$.

Solution

(a) Let

$$Q = Q_0 a^t.$$

Substituting $t = 20$, $Q = 88.2$ and $t = 23$, $Q = 91.4$ gives two equations for Q_0 and a:

$$88.2 = Q_0 a^{20} \quad \text{and} \quad 91.4 = Q_0 a^{23}.$$

Dividing the two equations enables us to eliminate Q_0:

$$\frac{91.4}{88.2} = \frac{Q_0 a^{23}}{Q_0 a^{20}} = a^3.$$

Solving for the base, a, gives

$$a = \left(\frac{91.4}{88.2}\right)^{1/3} = 1.012.$$

(b) Since $a = 1.012$, the growth rate is $0.012 = 1.2\%$.

(c) We want to evaluate $f(25) = Q_0a^{25} = Q_0(1.012)^{25}$. First we find Q_0 from the equation

$$88.2 = Q_0(1.012)^{20}.$$

Solving gives $Q_0 = 69.5$. Thus,

$$f(25) = 69.5(1.012)^{25} = 93.6.$$

Half-Life and Doubling Time

Radioactive substances, such as uranium, decay exponentially. A certain percentage of the mass disintegrates in a given unit of time; the time it takes for half the mass to decay is called the *half-life* of the substance.

A well-known radioactive substance is carbon-14, which is used to date organic objects. When a piece of wood or bone was part of a living organism, it accumulated small amounts of radioactive carbon-14. Once the organism dies, it no longer picks up carbon-14. Using the half-life of carbon-14 (about 5730 years), we can estimate the age of the object. We use the following definitions:

The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.

The Family of Exponential Functions

The formula $P = P_0a^t$ gives a family of exponential functions with positive parameters P_0 (the initial quantity) and a (the base, or growth/decay factor). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.19 and 1.20.) All members of the family $P = P_0 a^t$ are concave up.

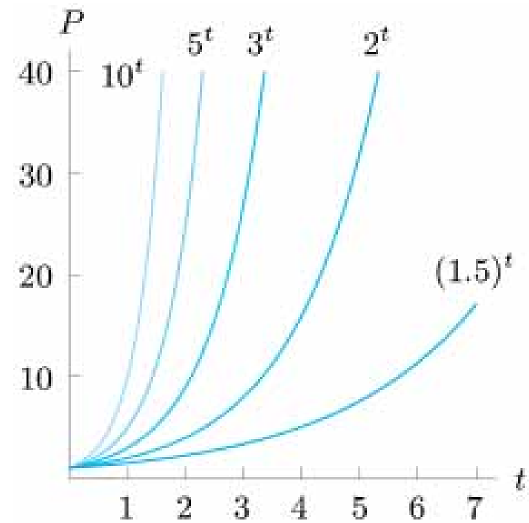


Figure 1.19 Exponential growth: $P = a^t$, for $a > 1$

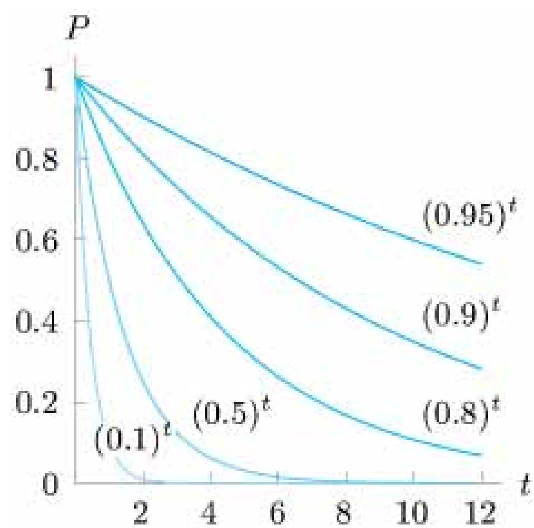


Figure 1.20 Exponential decay: $P = a^t$, for $0 < a < 1$

Example 2

Figure 1.21 is the graph of three exponential functions. What can you say about the values of the six constants, a, b, c, d, p, q?

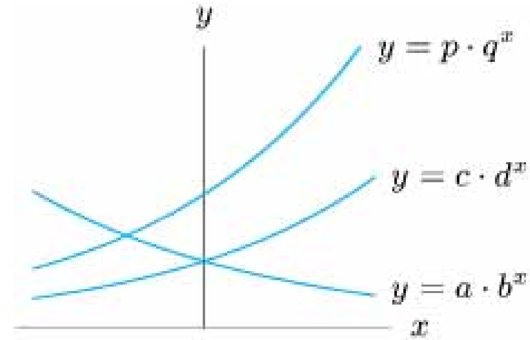


Figure 1.21

Solution

All the constants are positive. Since a, c, p represent y-intercepts, we see that $a = c$ because these graphs intersect on the y-axis. In addition, $a = c < p$, since $y = p \cdot q^x$ crosses the y-axis above the other two.

Since $y = a \cdot b^x$ is decreasing, we have $0 < b < 1$. The other functions are increasing, so $1 < d$ and $1 < q$.

Exponential Functions with Base *e*

The most frequently used base for an exponential function is the famous number $e = 2.71828\ldots$. This base is used so often that you will find an e^x button on most scientific calculators. At first glance, this is all somewhat mysterious. Why is it convenient to use the base 2.71828…? The full answer to that question must wait until Chapter 3, where we show that many calculus formulas come out neatly when e is used as the base. We often use the following result:

Any **exponential growth** function can be written, for some $a > 1$ and $k > 0$, in the form

$$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}$$

and any **exponential decay** function can be written, for some $0 < a < 1$ and $k > 0$, as

$$Q = Q_0 a^t \quad \text{or} \quad Q = Q_0 e^{-kt},$$

where P_0 and Q_0 are the initial quantities.

We say that P and Q are growing or decaying at a *continuous*[5] *rate* of k. (For example, $k = 0.02$ corresponds to a continuous rate of 2%.)

Example 3

Convert the functions $P = e^{0.5t}$ and $Q = 5e^{-0.2t}$ into the form $y = y_0 a^t$. Use the results to explain the shape of the graphs in Figures 1.22 and 1.23.

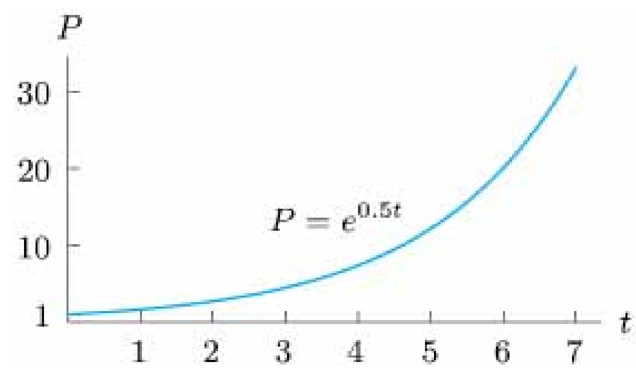


Figure 1.22 An exponential growth function

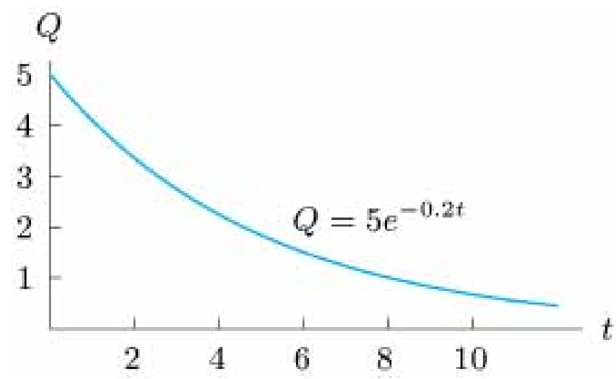


Figure 1.23 An exponential decay function

Solution

We have

$$P = e^{0.5t} = (e^{0.5})^t = (1.65)^t.$$

Thus, P is an exponential growth function with $P_0 = 1$ and $a = 1.65$. The function is increasing and its graph is concave up, similar to those in Figure 1.19. Also,

$$Q = 5e^{-0.2t} = 5(e^{-0.2})^t = 5(0.819)^t,$$

so Q is an exponential decay function with $Q_0 = 5$ and $a = 0.819$. The function is decreasing and its graph is concave up, similar to those in Figure 1.20.

Example 4

The quantity, Q, of a drug in a patient's body at time t is represented for positive constants S and k by the function $Q = S(1 - e^{-kt})$. For $t \geq 0$, describe how Q changes with time. What does S represent?

Solution

The graph of Q is shown in Figure 1.24. Initially none of the drug is present, but the quantity increases with time. Since the graph is concave down, the quantity increases at a decreasing rate. This is realistic because as the quantity of the drug in the body increases, so does the rate at which the body excretes the drug. Thus, we expect the quantity to level off. Figure 1.24 shows that S is the saturation level. The line $Q = S$ is called a *horizontal asymptote*.

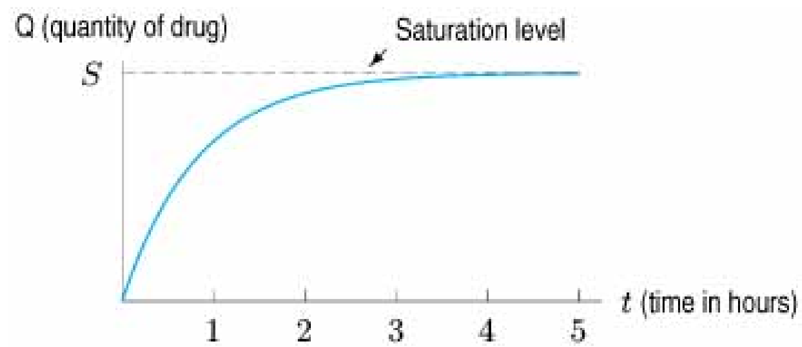


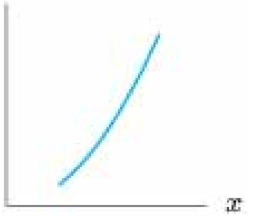
Figure 1.24 Buildup of the quantity of a drug in body

Exercises and Problems for Section 1.2

Exercises

In Exercises 1, 2, 3 and 4, decide whether the graph is concave up, concave down, or neither.

1.



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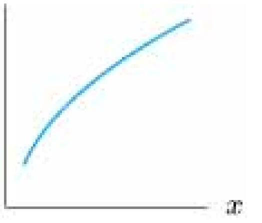

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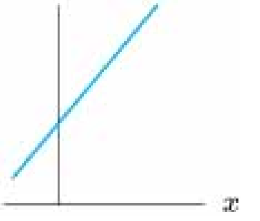

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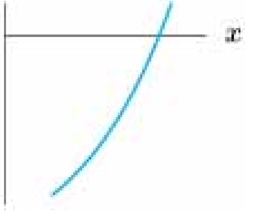


The functions in Exercises 5, 6, 7 and 8 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

5. $P = 5(1.07)^t$

6. $P = 7.7(0.92)^t$

7. $P = 3.2e^{0.03t}$

8. $P = 15e^{-0.06t}$

Write the functions in Problems 9, 10, 11 and 12 in the form $P = P_0a^t$. Which represent exponential growth and which represent exponential decay?

9. $P = 15e^{0.25t}$

10. $P = 2e^{-0.5t}$

11. $P = P_0e^{0.2t}$

12. $P = 7e^{-\pi t}$

In Problems 13 and 14, let $f(t) = Q_0a^t = Q_0(1 + r)^t$.

(a) Find the base, a.

(b) Find the percentage growth rate, r.

13. $f(5) = 75.94$ and $f(7) = 170.86$

14. $f(0.02) = 25.02$ and $f(0.05) = 25.06$

15. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.

(a) The population increases by 50 people a year.

(b) The population increases by 5% a year.

16. Identify the *x*-intervals on which the function graphed in Figure 1.25 is:

(a) Increasing and concave up

(b) Increasing and concave down

(c) Decreasing and concave up

(d) Decreasing and concave down

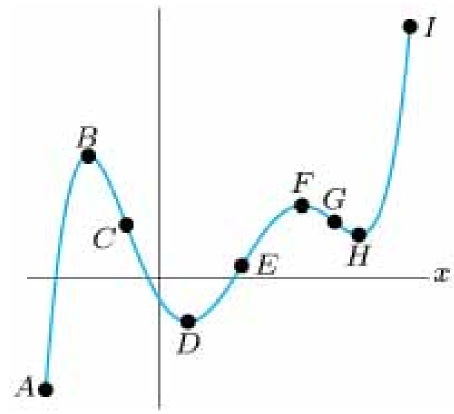


Figure 1.25

Problems

17. An air-freshener starts with 30 grams and evaporates. In each of the following cases, write a formula for the quantity, *Q* grams, of air-freshener remaining *t* days after the start and sketch a graph of the function. The decrease is:

(a) 2 grams a day

(b) 12% a day

18. In 2007, the world's population reached 6.7 billion and was increasing at a rate of 1.2% per year. Assume that this growth rate remains constant. (In fact, the growth rate has decreased since 1987.)

(a) Write a formula for the world population (in billions) as a function of the number of years since 2007.

(b) Use your formula to estimate the population of the world in the year 2020.

(c) Sketch a graph of world population as a function of years since 2007. Use the graph to estimate the doubling time of the population of the world.

19. A photocopy machine can reduce copies to 80% of their original size. By copying an already reduced copy, further reductions can be made.

(a) If a page is reduced to 80%, what percent enlargement is needed to return it to its original size?

(b) Estimate the number of times in succession that a page must be copied to make the final copy less than 15% of the size of the original.

20. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

(a) Graph the total number of people who have tried such a product against time.

(b) What do you know about the concavity of the graph?

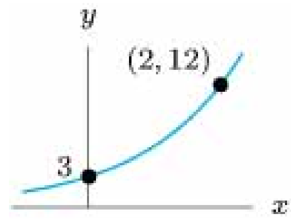
21. Sketch reasonable graphs for the following. Pay particular attention to the concavity of the graphs.

(a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.

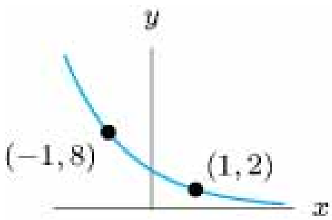
(b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.

Give a possible formula for the functions in Problems 22, 23, 24 and 25.

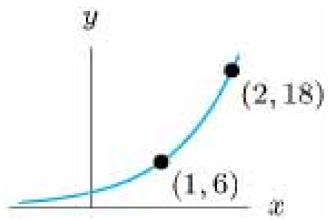
22.



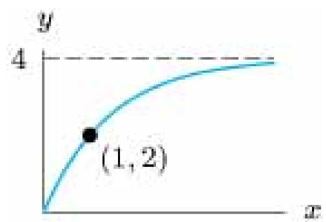
23.



24.



25.



26. (a) A population, P, grows at a continuous rate of 2% a year and starts at 1 million. Write P in the form $P = P_0e^{kt}$, with P_0, k constants.

(b) Plot the population in part (a) against time.

27. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially by 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

28. During April 2006, Zimbabwe's inflation rate averaged 0.67% a day. This means that, on average, prices went up by 0.67% from one day to the next.

(a) By what percentage did prices in Zimbabwe increase in April of 2006?

(b) Assuming the same rate all year, what was Zimbabwe's annual inflation rate during 2006?

29. (a) The half-life of radium-226 is 1620 years. Write a formula for the quantity, Q, of radium left after t years, if the initial quantity is Q_0.

(b) What percentage of the original amount of radium is left after 500 years?

30. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 1990?

31. A certain region has a population of 10,000,000 and an annual growth rate of 2%. Estimate the doubling time by guessing and checking.

32. Aircraft require longer takeoff distances, called takeoff rolls, at high altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0°C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

33. Each of the functions g, h, k in Table 1.8 is increasing, but each increases in a different way. Which of the graphs in Figure 1.26 best fits each function?

Table 1.8

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

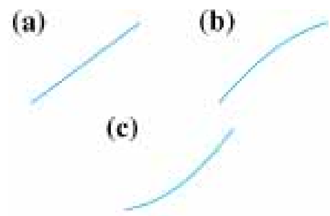


Figure 1.26

34. Each of the functions in Table 1.9 decreases, but each decreases in a different way. Which of the graphs in Figure 1.27 best fits each function?

Table 1.9

x	$f(x)$	$g(x)$	$h(x)$
1	100	22.0	9.3
2	90	21.4	9.1
3	81	20.8	8.8
4	73	20.2	8.4
5	66	19.6	7.9
6	60	19.0	7.3

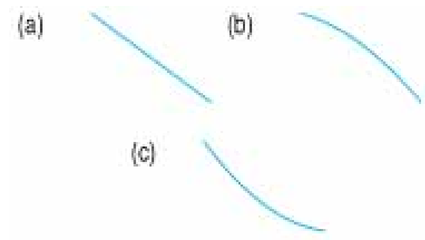


Figure 1.27

35. (a) Which (if any) of the functions in the following table could be linear? Find formulas for those functions.

(b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

36. The median price, P, of a home rose from \$60,000 in 1980 to \$180,000 in 2000. Let t be the number of years since 1980.

(a) Assume the increase in housing prices has been linear. Give an equation for the line representing price, P, in terms of t. Use this equation to complete column (a) of Table 1.10. Use units of \$1000.

(b) If instead the housing prices have been rising exponentially, find an equation of the form $P = P_0 a^t$ to represent housing prices. Complete column (b) of Table 1.10.

(c) On the same set of axes, sketch the functions represented in column (a) and column (b) of Table 1.10.

(d) Which model for the price growth do you think is more realistic?

Table 1.10

t	(a) Linear growth price in \$1000 units	(b) Exponential growth price in \$1000 units
0	60	60
10		
20	180	180
30		
40		

37. Estimate graphically the doubling time of the exponentially growing population shown in Figure 1.28. Check that the doubling time is independent of where you start on the graph. Show algebraically that if $P = P_0 a^t$ doubles between time t and time $t + d$, then d is the same number for any t.

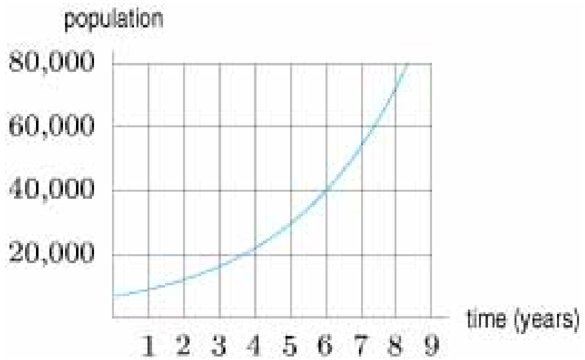


Figure 1.28