1.3 New Functions from Old

Shifts and Stretches

The graph of a constant multiple of a given function is easy to visualize: each y-value is stretched or shrunk by that multiple. For example, consider the function $f(x)$ and its multiples $y = 3f(x)$ and $y = -2f(x)$. Their graphs are shown in Figure 1.29. The factor 3 in the function $y = 3f(x)$ stretches each $f(x)$ value by multiplying it by 3; the factor -2 in the function $y = -2f(x)$ stretches $f(x)$ by multiplying by 2 and reflects it about the x-axis. You can think of the multiples of a given function as a family of functions.

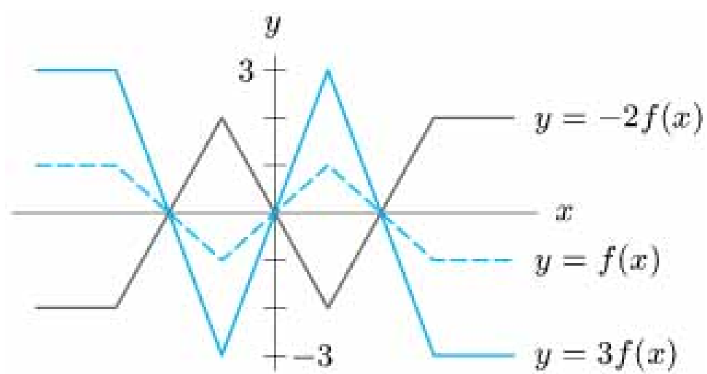


Figure 1.29 Multiples of the function $f(x)$

It is also easy to create families of functions by shifting graphs. For example, $y - 4 = x^2$ is the same as

$y = x^2 + 4$, which is the graph of $y = x^2$ shifted up by 4. Similarly, $y = (x - 2)^2$ is the graph of $y = x^2$ shifted right by 2. (See Figure 1.30.)

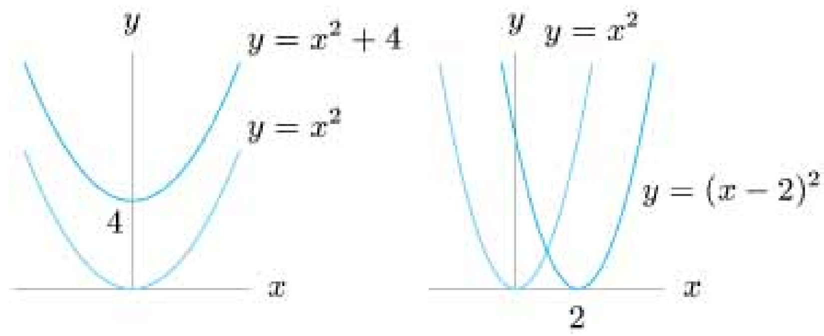


Figure 1.30 Graphs of $y = x^2$ with $y = x^2 + 4$ and $y = (x - 2)^2$

- Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph about the x-axis, in addition to shrinking or stretching.
- Replacing y by $(y - k)$ moves a graph up by k (down if k is negative).
- Replacing x by $(x - h)$ moves a graph to the right by h (to the left if h is negative).

Composite Functions

If oil is spilled from a tanker, the area of the oil slick grows with time. Suppose that the oil slick is always a perfect circle. Then the area, A, of the oil slick is a function of its radius, r:

$$A = f(r) = \pi r^2.$$

The radius is also a function of time, because the radius increases as more oil spills. Thus, the area, being a function of the radius, is also a function of time. If, for example, the radius is given by

$$r = g(t) = 1 + t,$$

then the area is given as a function of time by substitution:

$$A = \pi r^2 = \pi(1 + t)^2.$$

We are thinking of A as a *composite function* or a "function of a function," which is written

$$A = \underbrace{f(g(t))}_{\substack{\text{Composite function;} \\ f \text{ is outside function,} \\ g \text{ is inside function}}} = \pi(g(t))^2 = \pi(1+t)^2.$$

To calculate A using the formula $\pi(1 + t)^2$, the first step is to find $1 + t$, and the second step is to square and multiply by π. The first step corresponds to the inside function $g(t) = 1 + t$, and the second step corresponds to the outside function $f(r) = \pi r^2$.

Example 1

If $f(x) = x^2$ and $g(x) = x + 1$, find each of the following:

(a) $f(g(2))$
(b) $g(f(2))$
(c) $f(g(x))$
(d) $g(f(x))$

Solution

(a) Since $g(2) = 3$, we have $f(g(2)) = f(3) = 9$.
(b) Since $f(2) = 4$, we have $g(f(2)) = g(4) = 5$. Notice that $f(g(2)) \neq g(f(2))$.
(c) $f(g(x)) = f(x + 1) = (x + 1)^2$.
(d) $g(f(x)) = g(x^2) = x^2 + 1$. Again, notice that $f(g(x)) \neq g(f(x))$.

Example 2

Express each of the following functions as a composition:

(a) $h(t) = (1 + t^3)^{27}$

(b) $k(y) = e^{-y^2}$

(c) $l(y) = -(e^y)^2$

Solution

In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

(a) For $(1 + t^3)^{27}$, the first stage is cubing and adding 1, so an inside function is $g(t) = 1 + t^3$. The second stage is taking the 27^{th} power, so an outside function is $f(y) = y^{27}$. Then

$$f(g(t)) = f(1+t^3) = (1+t^3)^{27}.$$

In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1 + y)^{27}$ is another possibility.

(b) To calculate e^{-y^2} we square y, take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

$$f(g(y)) = e^{-y^2}.$$

(c) To calculate $-(e^y)^2$, we find e^y, square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

$$g(f(y)) = -(e^y)^2.$$

Since parts (b) and (c) give different answers, we see the order in which functions are composed is important.

Odd and Even Functions: Symmetry

There is a certain symmetry apparent in the graphs of $f(x) = x^2$ and $g(x) = x^3$ in Figure 1.31. For each point (x, x^2) on the graph of f, the point $(-x, x^2)$ is also on the graph; for each point (x, x^3) on the graph of g, the point $(-x, -x^3)$ is also on the graph. The graph of $f(x) = x^2$ is symmetric about the y-axis, whereas the graph of $g(x) = x^3$ is symmetric about the origin. The graph of any polynomial involving only even powers of x has symmetry about the y-axis, while polynomials with only odd powers of x are symmetric about the origin. Consequently, any functions with these symmetry properties are called *even* and *odd*, respectively.

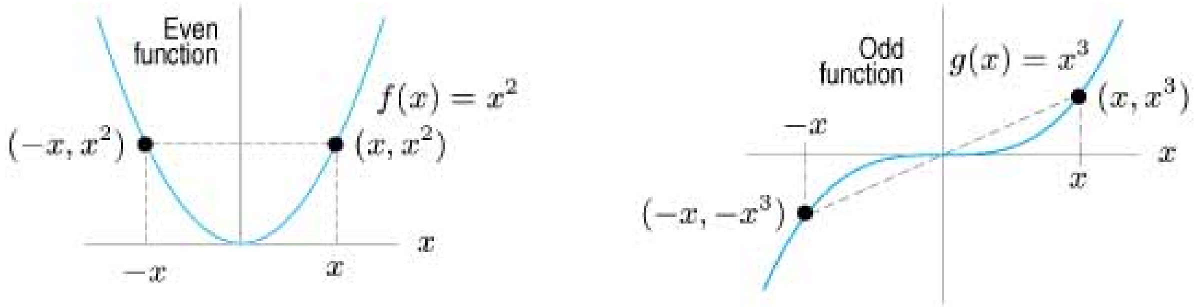


Figure 1.31 Symmetry of even and odd functions

For any function f,

f is an **even** function if $f(-x) = f(x)$ for all x.
f is an **odd** function if $f(-x) = -f(x)$ for all x.

For example, $g(x) = e^{x^2}$ is even and $h(x) = x^{1/3}$ is odd. However, many functions do not have any symmetry and are neither even nor odd.

Inverse Functions

On August 26, 2005, the runner Kenenisa Bekele of Ethiopia set a world record for the 10,000-meter race. His times, in seconds, at 2000-meter intervals are recorded in Table 1.11, where $t = f(d)$ is the number of seconds Bekele took to complete the first d meters of the race. For example, Bekele ran the first 4000 meters in 629.98 seconds, so $f(4000) = 629.98$. The function f was useful to athletes planning to compete with Bekele.

Table 1.11 *Bekele's Running Time*

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	315.63
4000	629.98
6000	944.66
8000	1264.63
10000	1577.53

Let us now change our point of view and ask for distances rather than times. If we ask how far Bekele ran during the first 629.98 seconds of his race, the answer is clearly 4000 meters. Going backward in this way from numbers of seconds to numbers of meters gives f^{-1}, the *inverse function*[6] of f. We write $f^{-1}(629.98) = 4000$. Thus, $f^{-1}(t)$ is the number of meters that Bekele ran during the first t seconds of his race. See Table 1.12 which contains values of f^{-1}.

Table 1.12 *Distance Run by Bekele*

t (seconds)	$d = f^{-1}(t)$ (meters)
0.00	0
315.63	2000
629.98	4000
944.66	6000
1264.63	8000
1577.53	10000

The independent variable for f is the dependent variable for f^{-1}, and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \leq d \leq 10000$, which is the range of f^{-1}. The range of f is all times t, such that $0 \leq t \leq 1577.53$, which is the domain of f^{-1}.

Which Functions have Inverses?

If a function has an inverse, we say it is *invertible*. Let's look at a function which is not invertible. Consider the flight of the Mercury spacecraft *Freedom 7*, which carried Alan Shepard, Jr. into space in May 1961. Shepard was the first American to journey into space. After launch, his spacecraft rose to an altitude of 116 miles, and then came down into the sea. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why not, try to decide on a value for $f^{-1}(100)$, which should be the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.32.)

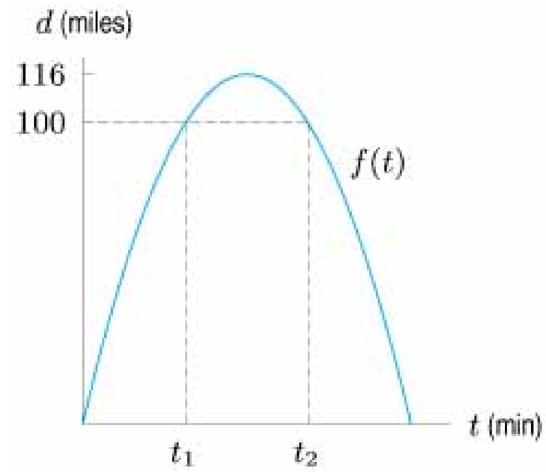


Figure 1.32 Two times, t_1 and t_2, at which altitude of spacecraft is 100 miles

The reason the altitude function does not have an inverse is that the altitude has the same value for two different times. The reason the Bekele time function did have an inverse is that each running time, t, corresponds to a unique distance, d.

Figure 1.33 suggests when an inverse exists. The original function, f, takes us from an x-value to a y-value, as shown in Figure 1.33. Since having an inverse means there is a function going from a y-value to an x-value, the crucial question is whether we can get back. In other words, does each y-value correspond to a unique x-value? If so, there's an inverse; if not, there is not. This principle may be stated geometrically, as follows:

> A function has an inverse if (and only if) its graph intersects any horizontal line at most once.

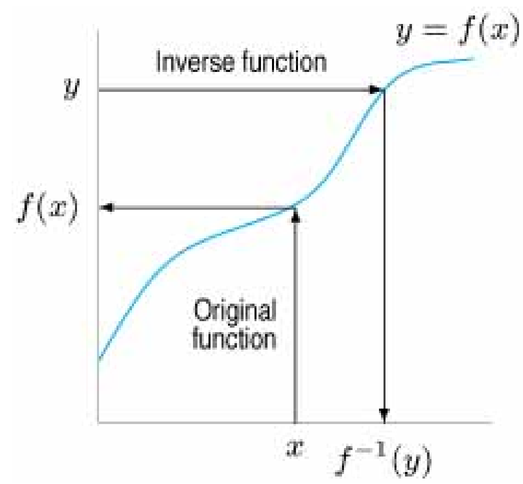


Figure 1.33 A function which has an inverse

For example, the function $f(x) = x^2$ does not have an inverse because many horizontal lines intersect the parabola twice.

Definition of an Inverse Function

If the function f is invertible, its inverse is defined as follows:

$$f^{-1}(y) = x \quad \text{means} \quad y = f(x).$$

Formulas for Inverse Functions

If a function is defined by a formula, it is sometimes possible to find a formula for the inverse function. In Section 1.1, we looked at the snow tree cricket, whose chirp rate, C, in chirps per minute, is approximated at the temperature, T, in degrees Fahrenheit, by the formula

$$C = f(T) = 4T - 160.$$

So far we have used this formula to predict the chirp rate from the temperature. But it is also possible to use this formula backward to calculate the temperature from the chirp rate.

Example 3

Find the formula for the function giving temperature in terms of the number of cricket chirps per minute; that is, find the inverse function f^{-1} such that

$$T = f^{-1}(C).$$

Solution

Since C is an increasing function, f is invertible. We know $C = 4T - 160$. We solve for T, giving

$$T = \frac{C}{4} + 40,$$

so

$$f^{-1}(C) = \frac{C}{4} + 40.$$

Graphs of Inverse Functions

The function $f(x) = x^3$ is increasing everywhere and so has an inverse. To find the inverse, we solve

$$y = x^3$$

for x, giving

$$x = y^{1/3}.$$

The inverse function is

$$f^{-1}(y) = y^{1/3}$$

or, if we want to call the independent variable x,

$$f^{-1}(x) = x^{1/3}.$$

The graphs of $y = x^3$ and $y = x^{1/3}$ are shown in Figure 1.34. Notice that these graphs are the reflections of one another about the line $y = x$. For example, (8, 2) is on the graph of $y = x^{1/3}$ because $2 = 8^{1/3}$, and

(2, 8) is on the graph of $y = x^3$ because $8 = 2^3$. The points (8, 2) and (2, 8) are reflections of one another about the line $y = x$. In general, if the x- and y-axes have the same scales:

The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

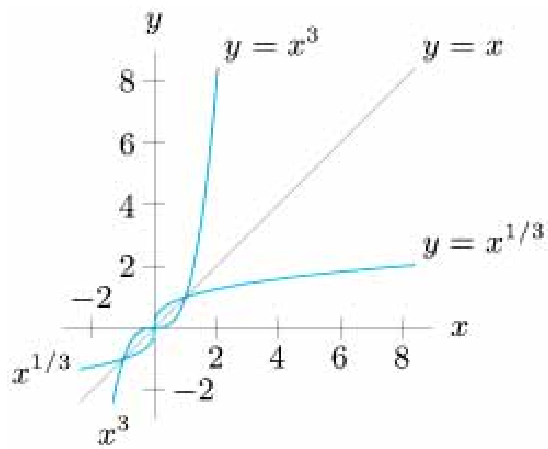


Figure 1.34 Graphs of inverse functions, $y = x^3$ and $y = x^{1/3}$, are reflections about the line $y = x$

Exercises and Problems for Section 1.3

Exercises

For the functions *f* in Exercises 1, 2 and 3, graph:

(a) $f(x+2)$

(b) $f(x-1)$

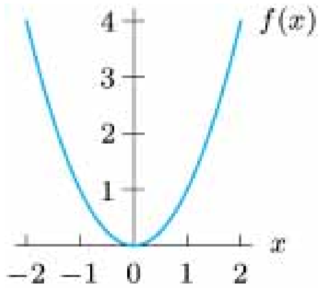
(c) $f(x)-4$

(d) $f(x+1)+3$

(e) $3f(x)$

(f) $-f(x)+1$

1.



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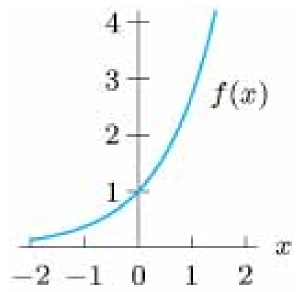

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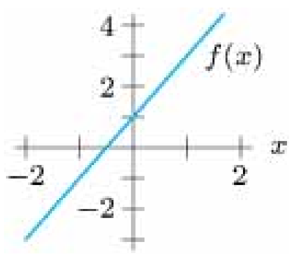


In Exercises 4, 5, 6 and 7, use Figure 1.35 to graph the functions.

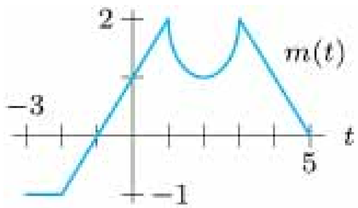


Figure 1.35

4. $n(t) = m(t) + 2$

5. $p(t) = m(t - 1)$

6. $k(t) = m(t + 1.5)$

7. $w(t) = m(t - 0.5) - 2.5$

For the functions f and g in Exercises 8, 9, 10 and 11, find

(a) $f(g(1))$

(b) $g(f(1))$

(c) $f(g(x))$

(d) $g(f(x))$

(e) $f(t)g(t)$

8. $f(x) = x^2$, $g(x) = x + 1$

9. $f(x) = \sqrt{x+4}, \; g(x) = x^2$

10. $f(x) = e^x$, $g(x) = x^2$

11. $f(x) = 1/x$, $g(x) = 3x + 4$

12. For $g(x) = x^2 + 2x + 3$, find and simplify:

(a) $g(2 + h)$

(b) $g(2)$

(c) $g(2 + h) - g(2)$

13. If $f(x) = x^2 + 1$, find and simplify:

(a) $f(t + 1)$

(b) $f(t^2 + 1)$

(c) $f(2)$

(d) $2f(t)$

(e) $[f(t)]^2 + 1$

14. For $f(n) = 3n^2 - 2$ and $g(n) = n + 1$, find and simplify:

(a) $f(n) + g(n)$

(b) $f(n)g(n)$

(c) The domain of $f(n)/g(n)$

(d) $f(g(n))$

(e) $g(f(n))$

Simplify the quantities in Exercises 15, 16, 17 and 18 using $m(z) = z^2$.

15. $m(z + 1) - m(z)$

16. $m(z + h) - m(z)$

17. $m(z) - m(z - h)$

18. $m(z + h) - m(z - h)$

19. Let p be the price of an item and q be the number of items sold at that price, where $q = f(p)$. What do the following quantities mean in terms of prices and quantities sold?

(a) $f(25)$

(b) $f^{-1}(30)$

20. Let $C = f(A)$ be the cost, in dollars, of building a store of area A square feet. In terms of cost and square feet, what do the following quantities represent?

(a) $f(10{,}000)$

(b) $f^{-1}(20{,}000)$

21. Let $f(x)$ be the temperature (°F) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(75)$ in practical terms?

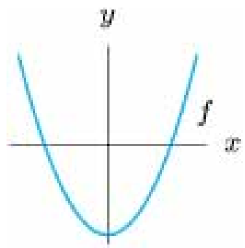
22. Let $m = f(A)$ be the minimum annual gross income, in thousands of dollars, needed to obtain a 30-year home mortgage loan of A thousand dollars at an interest rate of 6%. What do the following quantities represent in terms of the income needed for a loan?

(a) $f(100)$

(b) $f^{-1}(75)$

For Exercises 23 and 24, decide if the function $y = f(x)$ is invertible.

23.



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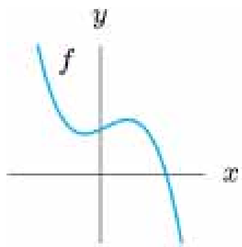


For Exercises 25, 26 and 27, use a graph of the function to decide whether or not it is invertible.

25. $f(x) = x^2 + 3x + 2$

26. $f(x) = x^3 - 5x + 10$

27. $f(x) = x^3 + 5x + 10$

Are the functions in Exercises 28, 29, 30, 31, 32, 33, 34 and 35 even, odd, or neither?

28. $f(x) = x^6 + x^3 + 1$

29. $f(x) = x^3 + x^2 + x$

30. $f(x) = x^4 - x^2 + 3$

31. $f(x) = x^3 + 1$

32. $f(x) = 2x$

33. $f(x) = e^{x^2-1}$

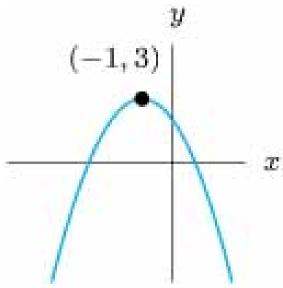
34. $f(x) = x(x^2 - 1)$

35. $f(x) = e^x - x$

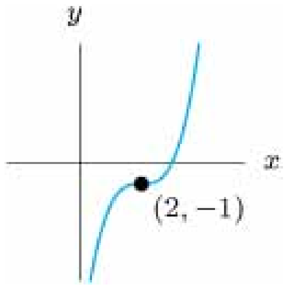
Problems

Find possible formulas for the graphs in Exercises 36 and 37 using shifts of x^2 or x^3.

36.



37.



38. (a) Use Figure 1.36 to estimate $f^{-1}(2)$.

(b) Sketch a graph of f^{-1} on the same axes.

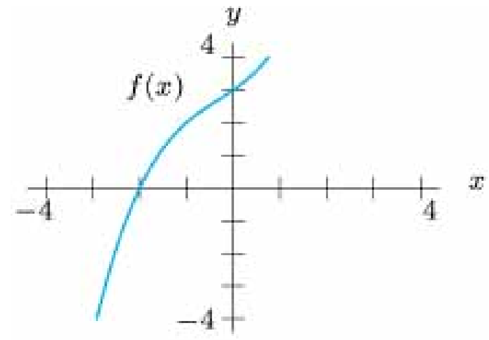


Figure 1.36

39. How does the graph of $Q = S(1 - e^{-kt})$ in Example 4 relate to the graph of the exponential decay function, $y = Se^{-kt}$?

40. Write a table of values for f^{-1}, where f is as given below. The domain of f is the integers from 1 to 7. State the domain of f^{-1}.

x	1	2	3	4	5	6	7
$f(x)$	3	-7	19	4	178	2	1

For Problems 41, 42, 43 and 44, decide if the function f is invertible.

41. $f(d)$ is the total number of gallons of fuel an airplane has used by the end of d minutes of a particular flight.

42. $f(t)$ is the number of customers in Macy's department store at t minutes past noon on December 18, 2008.

43. $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.

44. $f(w)$ is the cost of mailing a letter weighing w grams.

For Problems 45, 46, 47, 48, 49 and 50, use the graphs in Figure 1.37.

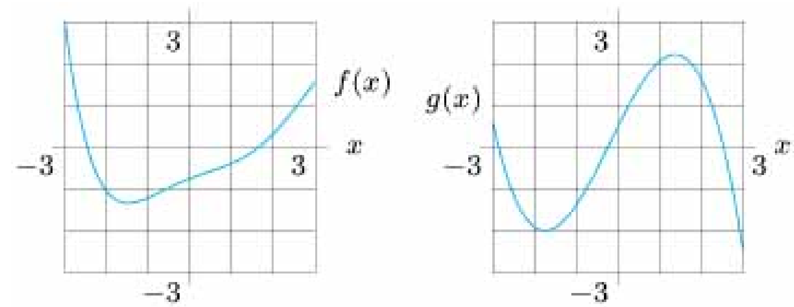


Figure 1.37

45. Estimate $f(g(1))$.

46. Estimate $g(f(2))$.

47. Estimate $f(f(1))$.

48. Graph $f(g(x))$.

49. Graph $g(f(x))$.

50. Graph $f(f(x))$.

In Problems 51, 52, 53 and 54, use Figure 1.38 to estimate the function value or explain why it cannot be done.

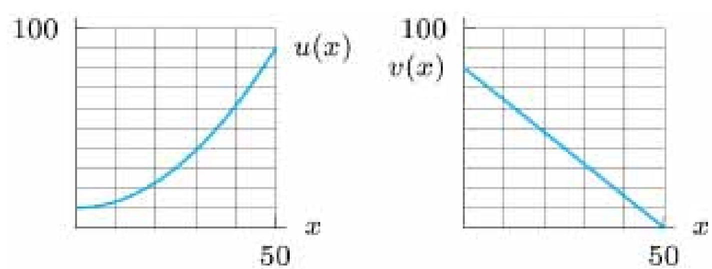


Figure 1.38

51. $u(v(10))$

52. $u(v(40))$

53. $v(u(10))$

54. $v(u(40))$

For Problems 55, 56, 57 and 58, determine functions f and g such that $h(x) = f(g(x))$. [Note: There is more than one correct answer. Do not choose $f(x) = x$ or $g(x) = x$.]

55. $h(x) = (x + 1)^3$

56. $h(x) = x^3 + 1$

57. $h(x) = \sqrt{x^2 + 4}$

58. $h(x) = e^{2x}$

59. A spherical balloon is growing with radius $r = 3t + 1$, in centimeters, for time t in seconds. Find the volume of the balloon at 3 seconds.

60. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves where $n = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.

61. The cost of producing q articles is given by the function $C = f(q) = 100 + 2q$.

(a) Find a formula for the inverse function.

(b) Explain in practical terms what the inverse function tells you.

62. A kilogram weighs about 2.2 pounds.

(a) Write a formula for the function, f, which gives an object's mass in kilograms, k, as a function of its weight in pounds, p.

(b) Find a formula for the inverse function of f. What does this inverse function tell you, in practical terms?

63. The graph of $f(x)$ is a parabola that opens upward and the graph of $g(x)$ is a line with negative slope. Describe the graph of $g(f(x))$ in words.

64. Figure 1.39 is a graph of the function $f(t)$. Here $f(t)$ is the depth in meters below the Atlantic Ocean floor where t million-year-old rock can be found.[7]

(a) Evaluate $f(15)$, and say what it means in practical terms.

(b) Is f invertible? Explain.

(c) Evaluate $f^{-1}(120)$, and say what it means in practical terms.

(d) Sketch a graph of f^{-1}.

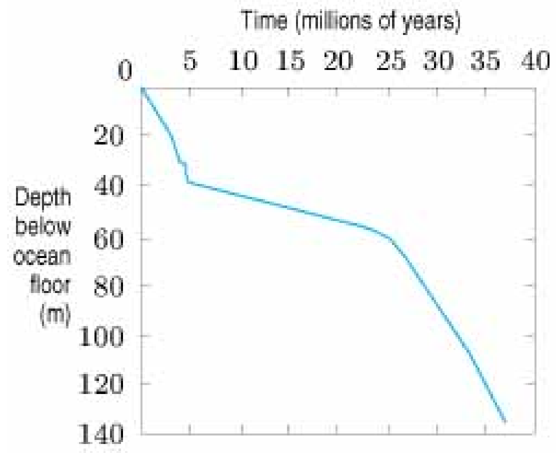


Figure 1.39