1.4 Logarithmic Functions

In Section 1.2, we approximated the population of Nevada (in millions) by the function

$$P = f(t) = 2.020(1.036)^t,$$

where *t* is the number of years since 2000. Now suppose that instead of calculating the population at time *t*, we ask when the population will reach 10 million. We want to find the value of *t* for which

$$10 = f(t) = 2.020(1.036)^t.$$

We use logarithms to solve for a variable in an exponent.

Logarithms to Base 10 and to Base e

We define the *logarithm* function, $\log_{10} x$, to be the inverse of the exponential function, 10^x, as follows:

> The **logarithm** to base 10 of *x*, written $\log_{10} x$, is the power of 10 we need to get *x*. In other words,
>
> $$\log_{10} x = c \quad \text{means} \quad 10^c = x.$$
>
> We often write $\log x$ in place of $\log_{10} x$.

The other frequently used base is *e*. The logarithm to base *e* is called the *natural logarithm* of *x*, written $\ln x$ and defined to be the inverse function of e^x, as follows:

> The **natural logarithm** of *x*, written $\ln x$, is the power of *e* needed to get *x*. In other words,
>
> $$\ln x = c \quad \text{means} \quad e^c = x.$$

Values of $\log x$ are in Table 1.13. Because no power of 10 gives 0, log 0 is undefined. The graph of $y = \log x$ is shown in Figure 1.40. The domain of $y = \log x$ is positive real numbers; the range is all real numbers. In contrast, the inverse function $y = 10^x$ has domain all real numbers and range all positive real numbers. The graph of $y = \log x$ has a vertical asymptote at $x = 0$, whereas $y = 10^x$ has a horizontal asymptote at $y = 0$.

Table 1.13 *Values for* $\log x$ *and* 10^x

x	$\log x$	x	10^x
0	undefined	0	1
1	0	1	10
2	0.3	2	100
3	0.5	3	10^3
4	0.6	4	10^4
⋮	⋮	⋮	⋮
10	1	10	10^{10}

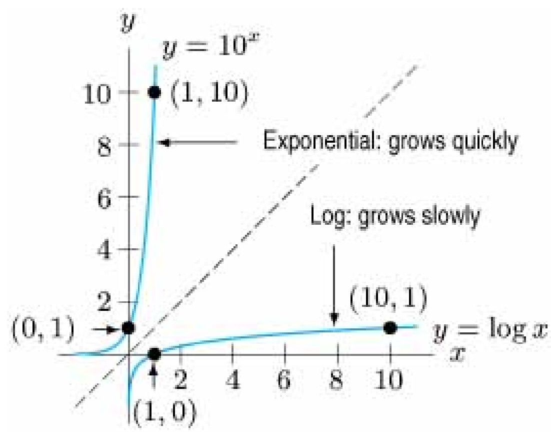

Figure 1.40 Graphs of $\log x$ and 10^x

One big difference between $y = 10^x$ and $y = \log x$ is that the exponential function grows extremely quickly whereas the log function grows extremely slowly. However, $\log x$ does go to infinity, albeit slowly, as x increases. Since $y = \log x$ and $y = 10^x$ are inverse functions, the graphs of the two functions are reflections of one another about the line $y = x$, provided the scales along the x- and y-axes are equal.

The graph of $y = \ln x$ in Figure 1.41 has roughly the same shape as the graph of $y = \log x$. The x-intercept is $x = 1$, since $\ln 1 = 0$. The graph of $y = \ln x$ also climbs very slowly as x increases. Both graphs, $y = \log x$ and $y = \ln x$, have *vertical asymptotes* at $x = 0$.

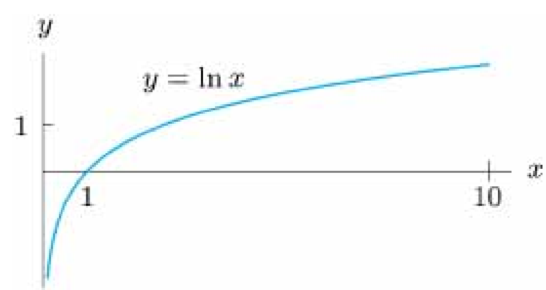

Figure 1.41 Graph of the natural logarithm

The following properties of logarithms may be deduced from the properties of exponents:

Properties of Logarithms

Note that $\log x$ and $\ln x$ are not defined when x is negative or 0.

1. $\log(AB) = \log A + \log B$	1. $\ln(AB) = \ln A + \ln B$
2. $\log\left(\frac{A}{B}\right) = \log A - \log B$	2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\log(A^p) = p \log A$	3. $\ln(A^p) = p \ln A$
4. $\log(10^x) = x$	4. $\ln e^x = x$
5. $10^{\log x} = x$	5. $e^{\ln x} = x$

In addition, $\log 1 = 0$ because $10^0 = 1$, and $\ln 1 = 0$ because $e^0 = 1$.

Solving Equations using Logarithms

Logs are frequently useful when we have to solve for unknown exponents, as in the next examples.

Example 1

Find t such that $2^t = 7$.

Solution

First, notice that we expect t to be between 2 and 3 (because $2^2 = 4$ and $2^3 = 8$). To calculate t, we take logs to base 10. (Natural logs could also be used.)

$$\log(2^t) = \log 7.$$

Then use the third property of logs, which says $\log(2^t) = t \log 2$, and get:

$$t \log 2 = \log 7.$$

Using a calculator to find the logs gives

$$t = \frac{\log 7}{\log 2} \approx 2.81.$$

Example 2

Find when the population of Nevada reaches 10 million by solving $10 = 2.020(1.036)^t$.

Solution

Dividing both sides of the equation by 2.020, we get

$$\frac{10}{2.020} = (1.036)^t.$$

Now take logs of both sides:

$$\log\left(\frac{10}{2.020}\right) = \log(1.036^t).$$

Using the fact that $\log(A^t) = t \log A$, we get

$$\log\left(\frac{10}{2.020}\right) = t \log(1.036).$$

Solving this equation using a calculator to find the logs, we get

$$t = \frac{\log(10 / 2.020)}{\log(1.036)} = 45.23 \text{ years}$$

which is between $t = 45$ and $t = 46$. This value of t corresponds to the year 2045.

Example 3

The release of chlorofluorocarbons used in air conditioners and in household sprays (hair spray, shaving cream, etc.) destroys the ozone in the upper atmosphere. Currently, the amount of ozone, Q, is decaying exponentially at a continuous rate of 0.25% per year. What is the half-life of ozone?

Solution

We want to find how long it takes for half the ozone to disappear. If Q_0 is the initial quantity of ozone, then

$$Q = Q_0 e^{-0.0025t}.$$

We want to find T, the value of t making $Q = Q_0/2$, that is,

$$Q_0 e^{-0.0025T} = \frac{Q_0}{2}.$$

Dividing by Q_0 and then taking natural logs yields

$$\ln(e^{-0.0025T}) = -0.0025T = \ln\left(\frac{1}{2}\right) \approx -0.6931,$$

so

$$T \approx 277 \text{ years}.$$

The half-life of ozone is about 277 years.

In Example 3 the decay rate was given. However, in many situations where we expect to find exponential growth or decay, the rate is not given. To find it, we must know the quantity at two different times and then solve for the growth or decay rate, as in the next example.

Example 4

The population of Mexico was 99.9 million in 2000 and 106.2 million in 2005. Assuming it increases exponentially, find a formula for the population of Mexico as a function of time.

Solution

If we measure the population, P, in millions and time, t, in years since 2000, we can say

$$P = P_0 e^{kt} = 99.9e^{kt},$$

where P_0 = 99.9 is the initial value of P. We find k by using the fact that P = 106.2 when t = 5, so

$$106.2 = 99.9e^{k \cdot 5}.$$

To find k, we divide both sides by 99.9, giving

$$\frac{106.2}{99.9} = 1.063 = e^{5k}.$$

Now take natural logs of both sides:

$$\ln(1.063) = \ln(e^{5k}).$$

Using a calculator and the fact that $\ln(e^{5k}) = 5k$, this becomes

$$0.061 = 5k.$$

So

$$k \approx 0.012,$$

and therefore

$$P = 99.9e^{0.012t}.$$

Since k = 0.012 = 1.2%, the population of Mexico was growing at a continuous rate of 1.2% per year.

In Example 4 we chose to use e for the base of the exponential function representing Mexico's population, making clear that the continuous growth rate was 1.2%. If we had wanted to emphasize the annual growth rate, we could have expressed the exponential function in the form $P = P_0 a^t$.

Example 5

Give a formula for the inverse of the following function (that is, solve for t in terms of P):

$$P = f(t) = 2.020(1.036)^t.$$

Solution

We want a formula expressing t as a function of P. Take logs:

$$\log P = \log(2.020(1.036)^t).$$

Since $\log(AB) = \log A + \log B$, we have

$$\log P = \log 2.020 + \log((1.036)^t).$$

Now use $\log(A^t) = t \log A$:

$$\log P = \log 2.020 + t \log 1.036.$$

Solve for t in two steps, using a calculator at the final stage:

$$t \log 1.036 = \log P - \log 2.020$$

$$t = \frac{\log P}{\log 1.036} - \frac{\log 2.020}{\log 1.036} = 65.11 \log P - 19.88.$$

Thus,

$$f^{-1}(P) = 65.11 \log P - 19.88.$$

Note that

$$f^{-1}(10) = 65.11(\log 10) - 19.88 = 65.11(1) - 19.88 = 45.23,$$

which agrees with the result of Example 2.

Exercises and Problems for Section 1.4

Exercises

Simplify the expressions in Exercises 1, 2, 3, 4, 5 and 6 completely.

1. $e^{\ln(1/2)}$

2. $10^{\log(AB)}$

3. $5e^{\ln(A^2)}$

4. $\ln(e^{2AB})$

5. $\ln(1/e) + \ln(AB)$

6. $2\ln(e^A) + 3\ln B^e$

For Exercises 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18, solve for x using logs.

7. $3^x = 11$

8. $17^x = 2$

9. $20 = 50(1.04)^x$

10. $4 \cdot 3^x = 7 \cdot 5^x$

11. $7 = 5e^{0.2x}$

12. $2^x = e^{x+1}$

13. $50 = 600e^{-0.4x}$

14. $2e^{3x} = 4e^{5x}$

15. $7^{x+2} = e^{17x}$

16. $10^{x+3} = 5e^{7-x}$

17. $2x - 1 = e^{\ln x^2}$

18. $4e^{2x-3} - 5 = e$

For Exercises 19, 20, 21, 22, 23 and 24, solve for *t*. Assume *a* and *b* are positive constants and *k* is nonzero.

19. $a = b^t$

20. $P = P_0 a^t$

21. $Q = Q_0 a^{nt}$

22. $P_0 a^t = Q_0 b^t$

23. $a = be^t$

24. $P = P_0 e^{kt}$

In Exercises 25, 26, 27 and 28, put the functions in the form $P = P_0e^{kt}$.

25. $P = 15(1.5)^t$

26. $P = 10(1.7)^t$

27. $P = 174(0.9)^t$

28. $P = 4(0.55)^t$

Find the inverse function in Exercises 29, 30 and 31.

29. $p(t) = (1.04)^t$

30. $f(t) = 50e^{0.1t}$

31. $f(t) = 1 + \ln t$

Problems

32. Without a calculator or computer, match the functions e^x, $\ln x$, x^2, and $x^{1/2}$ to their graphs in Figure 1.42.

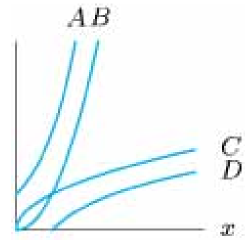

Figure 1.42

33. Find the equation of the line l in Figure 1.43.

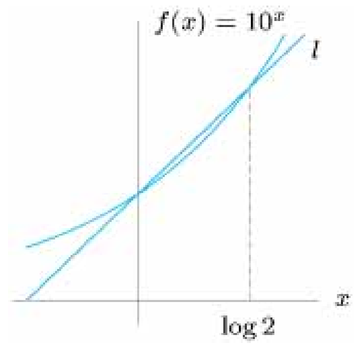

Figure 1.43

34. The exponential function $y(x) = Ce^{\alpha x}$ satisfies the conditions $y(0) = 2$ and $y(1) = 1$. Find the constants C and α. What is $y(2)$?

35. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on

(a) The y-intercept?

(b) The x-intercept?

36. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on

(a) The y-intercept?

(b) The x-intercept?

37. If $f(x) = a \ln(x + 2)$, what is the effect of increasing a on the vertical asymptote?

38. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on the vertical asymptote?

39. Is there a difference between $\ln[\ln(x)]$ and $\ln^2(x)$? [Note: $\ln^2(x)$ is another way of writing $(\ln x)^2$.]

40. The population of a region is growing exponentially. There were 40,000,000 people in 1990 ($t = 0$) and 56,000,000 in 2000. Find an expression for the population at any time t, in years. What population would you predict for the year 2010? What is the doubling time?

41. What is the doubling time of prices which are increasing by 5% a year?

42. The size of an exponentially growing bacteria colony doubles in 5 hours. How long will it take for the number of bacteria to triple?

43. One hundred kilograms of a radioactive substance decay to 40 kg in 10 years. How much remains after 20 years?

44. Find the half-life of a radioactive substance that is reduced by 30% in 20 hours.

45. The sales at Borders bookstores went from \$2108 million in 2000 to \$3880 million in 2005. Find an exponential function to model the sales as a function of years since 2000. What is the continuous percent growth rate, per year, of sales?

46. Owing to an innovative rural public health program, infant mortality in Senegal, West Africa, is being reduced at a rate of 10% per year. How long will it take for infant mortality to be reduced by 50%?

47. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A, in mg, remaining in the body is given by $A = 10(0.82)^t$.

(a) What was the initial amount taken?

(b) What percent of the drug leaves the body each hour?

(c) How much of the drug is left in the body 6 hours after the dose is administered?

(d) How long is it until only 1 mg of the drug remains in the body?

48. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.

(b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.

(c) Use logarithms to find the half-life of caffeine.

49. In 2000, there were about 213 million vehicles (cars and trucks) and about 281 million people in the US. The number of vehicles has been growing at 4% a year, while the population has been growing at 1% a year. If the growth rates remain constant, when is there, on average, one vehicle per person?

50. The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing according to the function $P = P_0e^{-kt}$, where t is time in hours. If 10% of the pollution is removed in the first five hours:

(a) What percentage of the pollution is left after 10 hours?

(b) How long is it before the pollution is reduced by 50%?

(c) Plot a graph of pollution against time. Show the results of your calculations on the graph.

(d) Explain why the quantity of pollutant might decrease in this way.

51. Air pressure, P, decreases exponentially with the height, h, in meters above sea level:

$$P = P_0e^{-0.00012h}$$

where P_0 is the air pressure at sea level.

(a) At the top of Mount McKinley, height 6194 meters (about 20,320 feet), what is the air pressure, as a percent of the pressure at sea level?

(b) The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?

52. The half-life of radioactive strontium-90 is 29 years. In 1960, radioactive strontium-90 was released into the atmosphere during testing of nuclear weapons, and was absorbed into people's bones. How many years does it take until only 10% of the original amount absorbed remains?

53. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.