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1.6 Powers, Polynomials, and Rational Functions

Power Functions

A *power function* is a function in which the dependent variable is proportional to a power of the independent variable:

A **power function** has the form

$$f(x) = kx^p, \quad \text{where } k \text{ and } p \text{ are constant.}$$

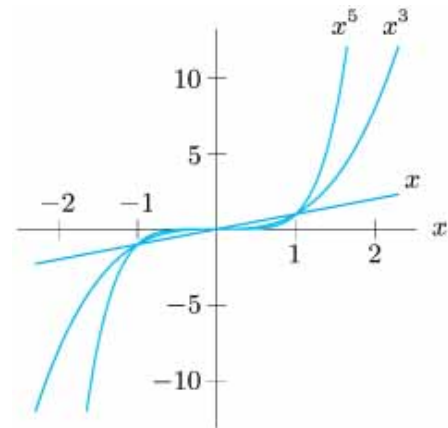
For example, the volume, V , of a sphere of radius r is given by


$$V = g(r) = \frac{4}{3} \pi r^3.$$

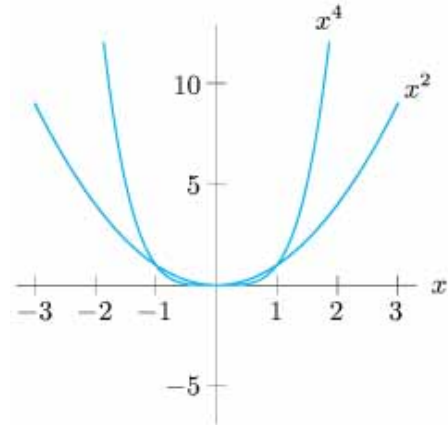
As another example, the gravitational force, F , on a unit mass at a distance r from the center of the earth is given by Newton's Law of Gravitation, which says that, for some positive constant k ,


$$F = \frac{k}{r^2} \quad \text{or} \quad F = kr^{-2}.$$

We consider the graphs of the power functions x^n , with n a positive integer. Figures [1.62](#) and [1.63](#) show that the graphs fall into two groups: odd and even powers. For n greater than 1, the odd powers have a “seat” at the origin and are increasing everywhere else. The even powers are first decreasing and then increasing. For large x , the higher the power of x , the faster the function climbs.



 **Figure 1.62** Odd powers of x : “Seat” shaped for $n > 1$

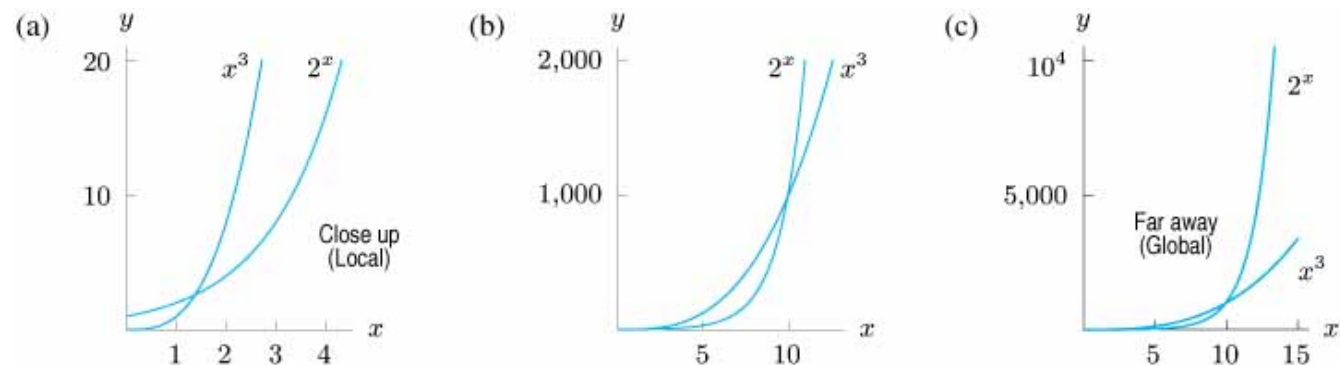



 **Figure 1.63** Even powers of x : U-shaped

Exponentials and Power Functions: Which Dominate?

In everyday language, the word exponential is often used to imply very fast growth. But do exponential functions always grow faster than power functions? To determine what happens “in the long run,” we often want to know which functions *dominate* as x gets arbitrarily large.

Let's consider $y = 2^x$ and $y = x^3$. The close-up view in Figure 1.64(a) shows that between $x = 2$ and $x = 4$, the graph of $y = 2^x$ lies below the graph of $y = x^3$. The far-away view in Figure 1.64(b) shows that the exponential function $y = 2^x$ eventually overtakes $y = x^3$. Figure 1.64(c), which gives a very far-away view, shows that, for large x , the value of x^3 is insignificant compared to 2^x . Indeed, 2^x is growing so much faster than x^3 that the graph of 2^x appears almost vertical in comparison to the more leisurely climb of x^3 .



 **Figure 1.64** Comparison of $y = 2^x$ and $y = x^3$: Notice that $y = 2^x$ eventually dominates $y = x^3$

We say that Figure 1.64(a) gives a *local* view of the functions' behavior, whereas Figure 1.64(c) gives a *global* view.

In fact, *every* exponential growth function eventually dominates *every* power function. Although an exponential function may be below a power function for some values of x , if we look at large enough x -values, a^x (with $a > 1$) will eventually dominate x^n , no matter what n is.

Polynomials

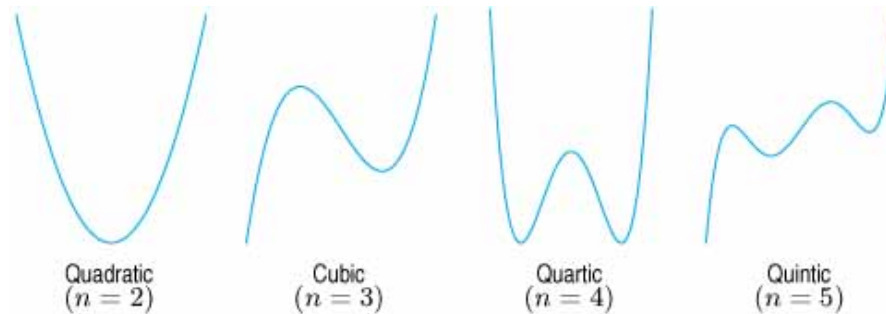
Polynomials are the sums of power functions with nonnegative integer exponents:


$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Here n is a nonnegative integer called the *degree* of the polynomial, and $a_n, a_{n-1}, \dots, a_1, a_0$ are constants, with leading coefficient $a_n \neq 0$. An example of a polynomial of degree $n = 3$ is

$$y = p(x) = 2x^3 - x^2 - 5x - 7.$$

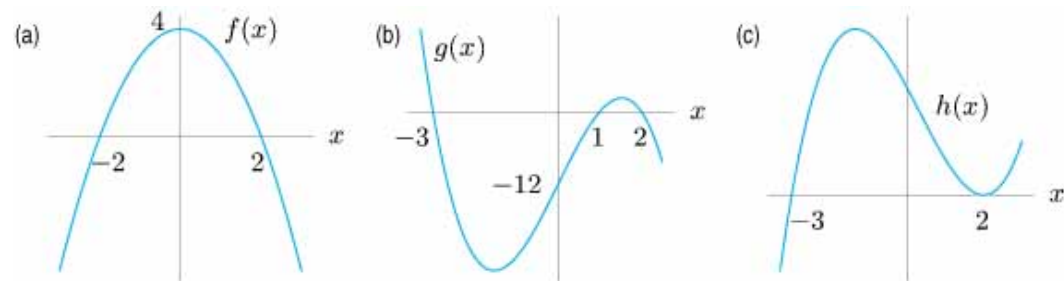
In this case $a_3 = 2$, $a_2 = -1$, $a_1 = -5$, and $a_0 = -7$. The shape of the graph of a polynomial depends on its degree; typical graphs are shown in Figure 1.65. These graphs correspond to a positive coefficient for x^n ; a negative leading coefficient turns the graph upside down. Notice that the quadratic “turns around” once, the cubic “turns around” twice, and the quartic (fourth degree) “turns around” three times. An n^{th} degree polynomial “turns around” at most $n - 1$ times (where n is a positive integer), but there may be fewer turns.



 **Figure 1.65** Graphs of typical polynomials of degree n

Example 1

Find possible formulas for the polynomials whose graphs are in Figure 1.66.



 **Figure 1.66** Graphs of polynomials

Solution

- (a) This graph appears to be a parabola, turned upside down, and moved up by 4, so

$$f(x) = -x^2 + 4.$$

The negative sign turns the parabola upside down and the +4 moves it up by 4. Notice that this formula does give the correct x -intercepts since $0 = -x^2 + 4$ has solutions $x = \pm 2$. These values of x are called *zeros* of f .

We can also solve this problem by looking at the x -intercepts first, which tell us that $f(x)$ has factors of $(x + 2)$ and $(x - 2)$. So

$$f(x) = k(x + 2)(x - 2).$$

To find k , use the fact that the graph has a y -intercept of 4, so $f(0) = 4$, giving

$$4 = k(0 + 2)(0 - 2),$$

or $k = -1$. Therefore, $f(x) = -(x + 2)(x - 2)$, which multiplies out to $-x^2 + 4$.

Note that $f(x) = 4 - x^4/4$ also has the same basic shape, but is flatter near $x = 0$. There are many possible answers to these questions.

- (b) This looks like a cubic with factors $(x + 3)$, $(x - 1)$, and $(x - 2)$, one for each intercept:

$$g(x) = k(x + 3)(x - 1)(x - 2).$$

Since the y -intercept is -12 , we have

$$-12 = k(0 + 3)(0 - 1)(0 - 2).$$

So $k = -2$, and we get the cubic polynomial

$$g(x) = -2(x + 3)(x - 1)(x - 2).$$

- (c) This also looks like a cubic with zeros at $x = 2$ and $x = -3$. Notice that at $x = 2$ the graph of $h(x)$ touches the x -axis but does not cross it, whereas at $x = -3$ the graph crosses the x -axis. We say that $x = 2$ is a *double zero*, but that $x = -3$ is a single zero.

To find a formula for $h(x)$, imagine the graph of $h(x)$ to be slightly lower down, so that the graph has one x -intercept near $x = -3$ and two near $x = 2$, say at $x = 1.9$ and $x = 2.1$. Then a formula would be

$$h(x) \approx k(x + 3)(x - 1.9)(x - 2.1).$$

Now move the graph back to its original position. The zeros at $x = 1.9$ and $x = 2.1$ move toward $x = 2$, giving

$$h(x) = k(x + 3)(x - 2)(x - 2) = k(x + 3)(x - 2)^2.$$

The double zero leads to a repeated factor, $(x - 2)^2$. Notice that when $x > 2$, the factor $(x - 2)^2$ is positive, and when $x < 2$, the factor $(x - 2)^2$ is still positive. This reflects the fact that $h(x)$ does not change sign near $x = 2$. Compare this with the behavior near the single zero at $x = -3$, where h does change sign.

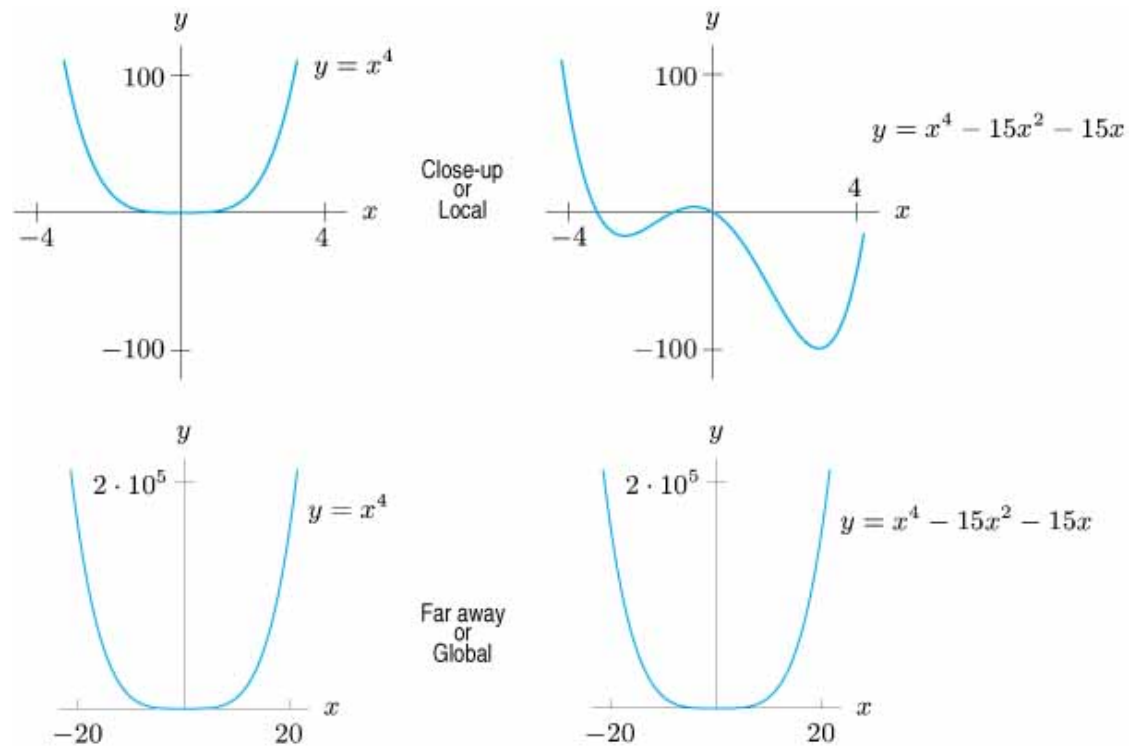
We cannot find k , as no coordinates are given for points off of the x -axis. Any positive value of k stretches the graph vertically but does not change the zeros, so any positive k works.


Example 2

Using a calculator or computer, graph $y = x^4$ and $y = x^4 - 15x^2 - 15x$ for $-4 \leq x \leq 4$ and for $-20 \leq x \leq 20$. Set the y range to $-100 \leq y \leq 100$ for the first domain, and to $-100 \leq y \leq 200,000$ for the second. What do you observe?

Solution

From the graphs in Figure 1.67 we see that close up ($-4 \leq x \leq 4$) the graphs look different; from far away, however, they are almost indistinguishable. The reason is that the leading terms (those with the highest power of x) are the same, namely x^4 , and for large values of x , the leading term dominates the other terms.



 **Figure 1.67** Local and global views of $y = x^4$ and $y = x^4 - 15x^2 - 15x$

Rational Functions

Rational functions are ratios of polynomials, p and q :

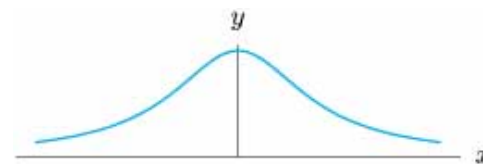
$$f(x) = \frac{p(x)}{q(x)}.$$

Example 3

Look at a graph and explain the behavior of $y = \frac{1}{x^2 + 4}$.

Solution

The function is even, so the graph is symmetric about the y -axis. As x gets larger, the denominator gets larger, making the value of the function closer to 0. Thus the graph gets arbitrarily close to the x -axis as x increases without bound. See Figure 1.68.



 **Figure 1.68** Graph of $y = \frac{1}{x^2 + 4}$

In the previous example, we say that $y = 0$ (i.e. the x -axis) is a *horizontal asymptote*. Writing “ \rightarrow ” to mean “tends to,” we have $y \rightarrow 0$ as $x \rightarrow \infty$ and $y \rightarrow 0$ as $x \rightarrow -\infty$.

If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$, then the line $y = L$ is called a **horizontal asymptote**.¹⁰ This occurs when

$$f(x) \rightarrow L \text{ as } x \rightarrow \infty \text{ or } f(x) \rightarrow L \text{ as } x \rightarrow -\infty.$$

If the graph of $y = f(x)$ approaches the vertical line $x = K$ as $x \rightarrow K$ from one side or the other, that is, if

$$y \rightarrow \infty \text{ or } y \rightarrow -\infty \text{ when } x \rightarrow K,$$

then the line $x = K$ is called a **vertical asymptote**.

The graphs of rational functions may have vertical asymptotes where the denominator is zero. For example, the function in Example 3 has no vertical asymptotes as the denominator is never zero. The function in Example 4 has two vertical asymptotes corresponding to the two zeros in the denominator.

Rational functions have horizontal asymptotes if $f(x)$ approaches a finite number as $x \rightarrow \infty$ or $x \rightarrow -\infty$. We call the behavior of a function as $x \rightarrow \pm\infty$ its *end behavior*.

Example 4

Look at a graph and explain the behavior of $y = \frac{3x^2 - 12}{x^2 - 1}$, including end behavior.

Solution

Factoring gives

$$y = \frac{3x^2 - 12}{x^2 - 1} = \frac{3(x+2)(x-2)}{(x+1)(x-1)}$$

so $x = \pm 1$ are vertical asymptotes. If $y = 0$, then $3(x+2)(x-2) = 0$ or $x = \pm 2$; these are the x -intercepts. Note that zeros of the denominator give rise to the vertical asymptotes, whereas zeros of the numerator give rise to x -intercepts.

Substituting $x = 0$ gives $y = 12$; this is the y -intercept. The function is even, so the graph is symmetric about the y -axis.

To see what happens as $x \rightarrow \pm\infty$, look at the y -values in Table 1.15. Clearly y is getting closer to 3 as x gets large positively or negatively. Alternatively, realize that as $x \rightarrow \pm\infty$, only the highest powers of x matter. For large x , the 12 and the 1 are insignificant compared to x^2 , so

$$y = \frac{3x^2 - 12}{x^2 - 1} \approx \frac{3x^2}{x^2} = 3 \quad \text{for large } x.$$

So $y \rightarrow 3$ as $x \rightarrow \pm\infty$, and therefore the horizontal asymptote is $y = 3$. See Figure 1.69. Since, for $x > 1$, the value of $(3x^2 - 12)/(x^2 - 1)$ is less than 3, the graph lies *below* its asymptote. (Why doesn't the graph lie below $y = 3$ when $-1 < x < 1$?)

Table 1.15 Values of

$$y = \frac{3x^2 - 12}{x^2 - 1}$$

x	$y = \frac{3x^2 - 12}{x^2 - 1}$
± 10	2.909091
± 100	2.999100
± 1000	2.999991

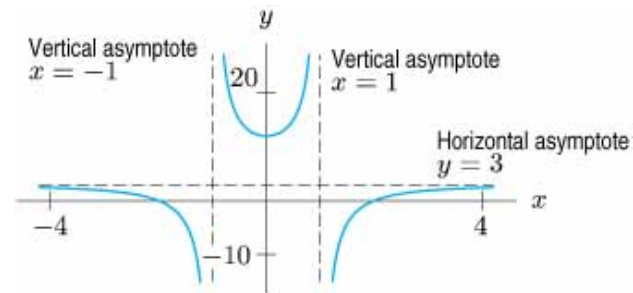


Figure 1.69

Graph of the function $y = \frac{3x^2 - 12}{x^2 - 1}$

Exercises and Problems for Section [1.6](#)

Exercises

For Problems [1](#) and [2](#), what happens to the value of the function as $x \rightarrow \infty$ and as $x \rightarrow -\infty$?

- $y = 0.25x^3 + 3$
- $y = 2 \cdot 10^{4x}$
- Determine the end behavior of each function as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.

(a) $f(x) = x^7$

(b) $f(x) = 3x + 7x^3 - 12x^4$

(c) $f(x) = x^{-4}$

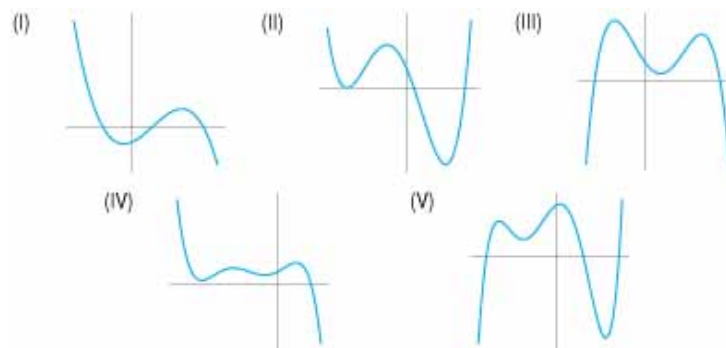
(d) $f(x) = \frac{6x^3 - 5x^2 + 2}{x^3 - 8}$

In Exercises [4](#) and [5](#), which function dominates as $x \rightarrow \infty$?

- $10 \cdot 2^x$ or $72,000x^{12}$
- $0.25\sqrt{x}$ or $25,000x^{-3}$

6. Each of the graphs in Figure 1.70 is of a polynomial. The windows are large enough to show end behavior.

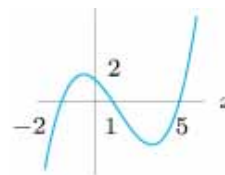
- (a) What is the minimum possible degree of the polynomial?
 (b) Is the leading coefficient of the polynomial positive or negative?



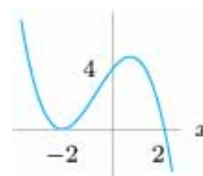
 **Figure 1.70**

Find cubic polynomials for the graphs in Exercises 7 and 8.

7.

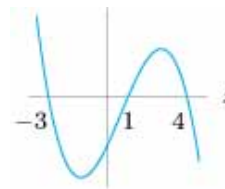


8.

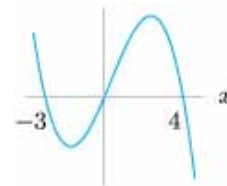


Find possible formulas for the graphs in Exercises 9, 10, 11 and 12.

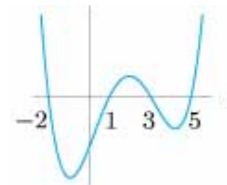
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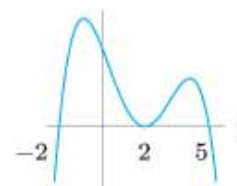
10.



11.



12.



Problems

13. Which of the functions I–III meet each of the following descriptions? There may be more than one function for each description, or none at all.

- (a) Horizontal asymptote of $y = 1$.
- (b) The x -axis is a horizontal asymptote.
- (c) Symmetric about the y -axis.
- (d) An odd function.
- (e) Vertical asymptotes at $x = \pm 1$.

I. $y = \frac{x-1}{x^2+1}$

II. $y = \frac{x^2-1}{x^2+1}$

III. $y = \frac{x^2+1}{x^2-1}$

14. The DuBois formula relates a person's surface area, s , in m^2 , to weight w , in kg, and height h , in cm, by

$$s = 0.01w^{0.25}h^{0.75}.$$

- (a) What is the surface area of a person who weighs 65 kg and is 160 cm tall?
 (b) What is the weight of a person whose height is 180 cm and who has a surface area of 1.5 m^2 ?
 (c) For people of fixed weight 70 kg, solve for h as a function of s . Simplify your answer.
15. A box of fixed volume V has a square base with side length x . Write a formula for the height, h , of the box in terms of x and V . Sketch a graph of h versus x .
16. According to *Car and Driver*, an Alfa Romeo going at 70 mph requires 177 feet to stop. Assuming that the stopping distance is proportional to the square of velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph (its top speed).
17. Water is flowing down a cylindrical pipe of radius r .
- (a) Write a formula for the volume, V , of water that emerges from the end of the pipe in one second if the water is flowing at a rate of
- (i) 3 cm/ sec
 (ii) k cm/ sec
- (b) Graph your answer to part (a)(ii) as a function of
- (i) r , assuming k is constant
 (ii) k , assuming r is constant
18. Poiseuille's Law gives the rate of flow, R , of a gas through a cylindrical pipe in terms of the radius of the pipe, r , for a fixed drop in pressure between the two ends of the pipe.
- (a) Find a formula for Poiseuille's Law, given that the rate of flow is proportional to the fourth power of the radius.
 (b) If $R = 400 \text{ cm}^3/\text{sec}$ in a pipe of radius 3 cm for a certain gas, find a formula for the rate of flow of that gas through a pipe of radius r cm.
 (c) What is the rate of flow of the same gas through a pipe with a 5 cm radius?
19. The height of an object above the ground at time t is given by

$$s = v_0t - \frac{g}{2}t^2,$$

where v_0 is the initial velocity and g is the acceleration due to gravity.

- (a) At what height is the object initially?
 (b) How long is the object in the air before it hits the ground?
 (c) When will the object reach its maximum height?
 (d) What is that maximum height?

20. A pomegranate is thrown from ground level straight up into the air at time $t = 0$ with velocity 64 feet per second. Its height at time t seconds is $f(t) = -16t^2 + 64t$. Find the time it hits the ground and the time it reaches its highest point. What is the maximum height?
21. (a) If $f(x) = ax^2 + bx + c$, what can you say about the values of a , b , and c if:
- (1, 1) is on the graph of $f(x)$?
 - (1, 1) is the vertex of the graph of $f(x)$? [Hint: The axis of symmetry is $x = -b/(2a)$.]
 - The y intercept of the graph is (0, 6)?
- (b) Find a quadratic function satisfying all three conditions.
22. Values of three functions are given in Table 1.16, rounded to two decimal places. One function is of the form $y = ab^t$, one is of the form $y = ct^2$, and one is of the form $y = kt^3$. Which function is which?

Table 1.16

t	$f(t)$	t	$g(t)$	t	$h(t)$
2.0	4.40	1.0	3.00	0.0	2.04
2.2	5.32	1.2	5.18	1.0	3.06
2.4	6.34	1.4	8.23	2.0	4.59
2.6	7.44	1.6	12.29	3.0	6.89
2.8	8.62	1.8	17.50	4.0	10.33
3.0	9.90	2.0	24.00	5.0	15.49

23. Values of three functions are given in Table 1.17, rounded to two decimal places. Two are power functions and one is an exponential. One of the power functions is a quadratic and one a cubic. Which one is exponential? Which one is quadratic? Which one is cubic?

Table 1.17

x	$f(x)$	x	$g(x)$	x	$k(x)$
8.4	5.93	5.0	3.12	0.6	3.24
9.0	7.29	5.5	3.74	1.0	9.01
9.6	8.85	6.0	4.49	1.4	17.66
10.2	10.61	6.5	5.39	1.8	29.19
10.8	12.60	7.0	6.47	2.2	43.61
11.4	14.82	7.5	7.76	2.6	60.91

24. A cubic polynomial with positive leading coefficient is shown in Figure 1.71 for $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. What can be concluded about the total number of zeros of this function? What can you say about the location of each of the zeros? Explain.

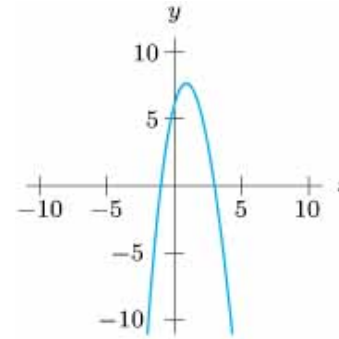


 Figure 1.71

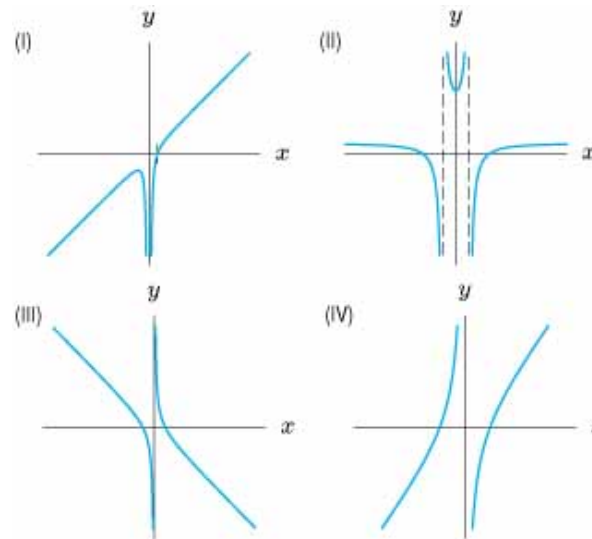
25. After running 3 miles at a speed of x mph, a man walked the next 6 miles at a speed that was 2 mph slower. Express the total time spent on the trip as a function of x . What horizontal and vertical asymptotes does the graph of this function have?
26. Match the following functions with the graphs in Figure 1.72. Assume $0 < b < a$.

(a) $y = \frac{a}{x} - x$

(b) $y = \frac{(x-a)(x+a)}{x}$

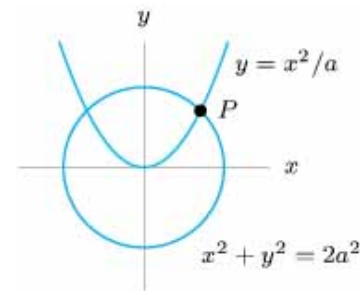
(c) $y = \frac{(x-a)(x^2+a)}{x^2}$

(d) $y = \frac{(x-a)(x+a)}{(x-b)(x+b)}$



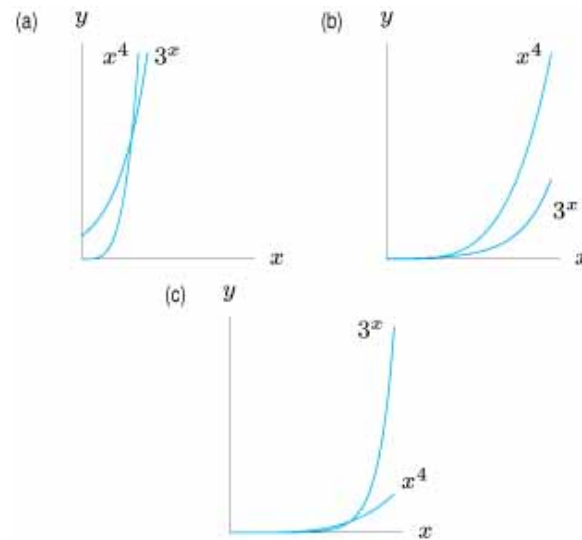
 **Figure 1.72**

27. Consider the point P at the intersection of the circle $x^2 + y^2 = 2a^2$ and the parabola $y = x^2/a$ in Figure 1.73. If a is increased, the point P traces out a curve. For $a > 0$, find the equation of this curve.



 **Figure 1.73**

28. Use a graphing calculator or a computer to graph $y = x^4$ and $y = 3^x$. Determine approximate domains and ranges that give each of the graphs in Figure 1.74.



 **Figure 1.74**

29. The rate, R , at which a population in a confined space increases is proportional to the product of the current population, P , and the difference between the carrying capacity, L , and the current population. (The carrying capacity is the maximum population the environment can sustain.)
- Write R as a function of P .
 - Sketch R as a function of P .

30. When an object of mass m moves with a velocity v that is small compared to the velocity of light, c , its energy is given approximately by

$$E \approx \frac{1}{2} mv^2.$$

If v is comparable in size to c , then the energy must be computed by the exact formula

$$E = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right).$$

- Plot a graph of both functions for E against v for $0 \leq v \leq 5 \cdot 10^8$ and $0 \leq E \leq 5 \cdot 10^{17}$. Take $m = 1$ kg and $c = 3 \cdot 10^8$ m/sec. Explain how you can predict from the exact formula the position of the vertical asymptote.
- What do the graphs tell you about the approximation? For what values of v does the first formula give a good approximation to E ?